FTPL WITH MONEY

1. FTPL WITH MONEY

This model is that of Sims (1994). Agent:

\[
\max_{\{C_t, M_t, B_t\}} E \left[ \sum_{t=0}^{\infty} \beta^t \log C_t \right] \quad \text{s.t.}
\]

\[
C_t(1 + \gamma f(v_t)) + \frac{M_t + B_t}{P_t} + \tau_t \leq \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} + Y_t
\]

\[
B_t \geq 0, \quad M_t \geq 0
\]

\[
v_t = \frac{P_tC_t}{M_t}.
\]

\(f\) is transactions costs as a proportion of total consumption. We assume \(f'(v) \geq 0\), all \(v > 0\), and \(f(0) = 0\). Additional conditions on \(f\) are needed to guarantee existence and uniqueness of the equilibrium under reasonable monetary and fiscal policies.

2. GOVERNMENT

GBC:

\[
\frac{B_t + M_t}{P_t} = \frac{R_{t-1}B_{t-1} + M_{t-1}}{P_t} - \tau_t
\]

Monetary policy:

\[
\begin{cases}
M_t \equiv \hat{M} \\
R_t \equiv \hat{R}
\end{cases}
\]

Fiscal policy:

\[
\begin{cases}
\tau_t = -\phi_0 + \phi_1 \frac{B_t}{P_t} \\
\bar{\tau} \equiv \bar{\tau}
\end{cases}
\]

Social Resource Constraint: From private constraint and GBC.

\[
C_t(1 + \gamma f(v_t)) = Y_t.
\]

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3. FOC’s

Assume an interior solution.

\[ \frac{1}{C_t} = \lambda_t (1 + \gamma f_t + \gamma' f_t v_t) \]

\[ \frac{\lambda_t}{P_t} = \beta R_t E_t \frac{\lambda_{t+1}}{P_{t+1}} \]

\[ \frac{\lambda_t}{P_t} (1 - \gamma f_t v_t^2) = \beta E_t \frac{\lambda_{t+1}}{P_{t+1}} . \]

The \( \partial B \) and \( \partial M \) conditions imply the “money demand” or “liquidity preference” relation

\[ 1 - \gamma f_t v_t^2 = R_t^{-1} . \]

4. Existence and Uniqueness

The \( \partial C \) and \( \partial M \) equations imply

\[ \frac{1 - \gamma f_t v_t^2}{P_t C_t (1 + \gamma f_t + \gamma' f_t v_t^2)} = \beta E_t \left[ \frac{1}{P_{t+1} C_{t+1} (1 + \gamma f_{t+1} + \gamma' f_{t+1} v_{t+1}^2)} \right] , \]

or, with \( Z_t = \frac{M_t}{P_t C_t (1 + \gamma f_t + \gamma' f_t v_t^2)} \),

\[ Z_t (1 - \gamma f_t v_t^2) = \beta E_t \left[ Z_{t+1} \frac{M_t}{M_{t+1}} \right] . \]  

\[ (*) \]

5. Case 1, \( M_t = \bar{M} \)

Then the \( M \) terms in the previous heading’s equation in \( Z \) cancel out.

Note that \( Z_t \) is a function \( g(\cdot) \) of \( v_t \) alone, and that for many (not all) “reasonable” \( f \)’s we can show that

(a) \( g'(v) < 0 \) for all \( v \);

(b) \( g(v) \xrightarrow{v \to 0} 0 \) and \( g(v) \xrightarrow{v \to \infty} \infty \).

6. Conditions for existence of steady state

If the model has a steady state with constant \( v_t = \bar{v} \), we will have, from \( (*) \),

\[ 1 - \gamma f'(\bar{v}) \bar{v}^2 = \beta . \]  

\[ (†) \]

If \( f(0) = 0 \), \( f \) absolutely continuous (i.e., differentiable except at a measure-zero set of points and equal to the integral of its derivative) and \( f'(v) \) is monotone near \( v = 0 \), then \( f'(v)v^2 \to 0 \) as \( v \to 0 \), even if \( f'(v) \to \infty \) as \( v \to 0 \). Existence of a stationary equilibrium is therefore determined by whether \( \gamma f'(v)v^2 \) can be as large as \( 1 - \beta \).
7. Two example $f$’s

Here and in what follows we will consider two example $f$’s:

$$f_t(v) = v$$
$$f_b(v) = \frac{v}{1+v}.$$ 

The linear $f_t$ implies that as real balances shrink to zero relative to consumption (so $v \to \infty$), consumption goes to zero at any fixed level of $Y$. The bounded $f_b$ implies that as real balances dwindle away, transactions costs approach some fixed fraction $\gamma/(1+\gamma) < 1$ of endowment.

We can see from (†) that for $f_t$, a steady state with constant $v$ always exists and that for $f_b$ it exists only if $\gamma \geq 1 - \beta$.

8. Uniqueness

To show this equilibrium is unique, we first show that, using $f_b$ or $f_t$, with any initial value of $Z_t$ above the steady state value $\bar{Z}$, $E_t[Z_{t+s}] \to \infty$ as $t \to \infty$, while with any initial value below $\bar{Z}$, $E_t[Z_{t+s}] \to 0$ as $t \to \infty$.

$Z_t$ is monotone decreasing in $v$ for both these $f$’s. (Prove this for yourself.)

∴ if $Z_t < \bar{Z}$, $E_t[Z_{t+1}] < \theta_t Z_t$ for some $\theta_t < 1$. This means that $P[Z_{t+1} \leq \theta_t Z_t \mid Z_t] > 0$. But then if $Z_{t+1} \leq Z_t$, we will have $E_t[Z_{t+2}] < \theta_t Z_{t+1}$ therefore that with non-zero conditional probability at $t+1$, $Z_{t+2} \leq \theta_t Z_{t+1}$, and therefore that with nonzero conditional probability at $t$, $Z_{t+2} \leq \theta_t^2 Z_t$. Continuing this argument recursively, we will have that, with non-zero conditional probability at $t$, $Z_{t+s} \leq \theta_t^s Z_t$. This implies that for every $\varepsilon > 0$, there is a non-zero probability that eventually $Z_t < \varepsilon$. But $v_t \to \infty$ as $Z_t \to 0$, so With non-zero probability $v_t$ becomes arbitrarily large if initially $Z_t < \bar{Z}$. A symmetric argument shows that if $Z_t > \bar{Z}$, $v_t$ gets arbitrarily close to zero with non-zero probability.

9. Can we rule out equilibria with arbitrarily large or small $Z$’s?

If we assume that $Y_t \geq \bar{Y}$ with probability one for all $t$, The only way we can have $v_t$ arbitrarily close to 0 with $M$ fixed is to have $P_t$ arbitrarily close to zero. This implies that $M_t/P_t$ must get arbitrarily large. Suppose the agent contemplates consuming a fraction $\delta$ of his real balances. As $M/P$ grows larger, the current utility gain from consuming this fraction of real balances gets arbitrarily large. If the agent contemplates keeping $M$ constant after the initial spending down of real balances, the resources available for consumption spending, $C(1 + \gamma f(v))$, will remain constant after the initial period. Since the agent will assume that his own actions have no effect on future prices, and since $C$ under this deviant decision rule will be if anything lower than on the original path, the effect on $v$ at later dates will be to increase it by no more than the ratio $1/(1-\delta)$. But with $P$ and $C(1 + \gamma f)$ held fixed, we can calculate

$$\frac{d\log C}{d\log M} = \frac{\gamma f'v}{1 + \gamma f}.$$
For either of our two example $f$’s, and indeed for any concave $f$ with $f(0) = 0$, this expression is bounded above by one. Thus the future utility costs generated by the reduction in $M$ by the factor $1 - \delta$, discounted to the current date, remain bounded, no matter how large is current $M/P$. But the current utility benefits of reducing $M$ by this factor become unboundedly large as $M/P$ increases. So it must be that the agent can increase expected utility by consuming some of his real balances if $M/P$ gets large enough.

REFERENCES