Empirical Implications of Arbitrage-Free Asset Markets

by

S. Maheswaran
Washington University, St. Louis

and

Christopher A. Sims
Yale University

January 1992

ABSTRACT

The martingale-equivalence condition delivered by a no-arbitrage assumption in complete asset markets has implications for fine-time-unit asset price behavior that can be rejected with finite spans of data. A class of stochastic processes that could model such deviations from martingale-equivalence is proposed.

Running head: Arbitrage-Free Asset Markets

Address correspondence and proofs to:

Christopher A. Sims
Dept. of Economics
Yale University
37 Hillhouse Ave.
New Haven, CT 06520-1962
Tel: 203 432 6292
Fax: 203 432 6323

Research for this paper was supported in part by NSF grant SES 8722451. The advice of referees and the editor has improved the paper, for which the authors are grateful.

EMPIRICAL IMPLICATIONS OF ARBITRAGE-FREE ASSET MARKETS

Price changes for a durable good with small storage costs must, in a frictionless competitive market, be in some sense unpredictable -- or so it seems intuitively. After all, if the good were reliably predicted to rise very rapidly in price, one would think the current price should be bid up by speculators eager to cash in on the predicted capital gains, while if it were reliably predicted to fall rapidly in price owners of the good would sell their holdings to avoid the predicted capital losses. These market reactions to predicted price rises or falls should prevent the occurrence of reliably predicted rises or falls. This intuitive idea has sometimes been formalized as the hypothesis that the price P_t of such a good should be a martingale relative to information observable by market participants, i.e. that, if X_t is data which becomes available at t,

$$E\left[P_{t+s} \,\middle|\, X_u,\ u \le t\right] = P_t, \tag{1}$$

for any $s > 0$.

But as emphasized by R.E. Lucas, Jr. [1978] and by Stephen F. Leroy [1973], among others, (1) emerges from such models only under extremely restrictive assumptions.

In a classic paper Harrison and Kreps [1979] showed that elimination of arbitrage opportunities would, in a market for assets traded continuously in time, force relative prices of assets to follow stochastic processes *equivalent* (in a technical measure-theoretic sense) to a martingale process. However this result is not in conflict with the findings of Lucas and Leroy that asset prices in equilibrium models do not in general follow martingale processes. The martingale-equivalence result is empty in a discrete-time model, because in such a model practically any stochastic process is equivalent to a martingale. Even in continuous time models the martingale equivalence result by itself places only weak restrictions on the observable behavior of asset prices.

If it is only the martingale equivalence property, not (1) (the martingale property itself), that is implied by theoretical analysis of arbitrage-free markets, then is it happenstance that econometric tests of (1) often show it to be close to correct? And is the intuitive notion that speculators should eliminate predictable price changes simply fallacious? We show that, with a continuous data record on prices, there are classes of behavior for prices that are inconsistent with martingale equivalence. Roughly speaking, we show that martingale equivalence restricts the nature of price behavior over fine time intervals, ruling out paths that show increasingly erratic behavior of rates of change over smaller and smaller time units if this erratic behavior can be to some extent predicted.

We attempt to make the paper readable at two levels. A reader should be able to understand the nature of our claims about the testable implications of arbitrage-freeness without having a firm enough grounding in the theory of continuous-time stochastic processes to understand our proofs. Thus in some sections of the paper we explain at an elementary level concepts necessary to understanding the claimed results, while in other parts of the paper we make arguments that assume a more mathematically sophisticated reader. While we refer to Jacod and Shiryayev [1987] for standard results, the treatment there is terse; readers approaching the subject for the first time might consult Protter [1990] or Dothan [1990].

1. Martingale Equivalence

In an appendix we state the technical definition of equivalence of probability measures and provide a concise derivation of the martingale equivalence result[1] . Here we characterize the result more intuitively. If we are given some events to which probabilities can be attached, we may consider two different probability distributions on them, μ and ν, each of which attaches a probability $\mu(S)$ or $\nu(S)$ to each event S. We say μ is *absolutely continuous* with respect to ν, or ν *dominates* μ, if every event with probability zero under ν has probability zero under μ, i.e. $\upsilon(S)=0 \Rightarrow \mu(S)=0$. If the implication goes the other way as well, so μ dominates ν also, then the two probabilities are *equivalent*. The events S are conditions which we may be able to observe. To say that μ and ν are equivalent is just to say that there is no event we could observe which would tell us with certainty that one or the other of the two probability models is correct. Such an event would be one which was impossible under one model (say $\mu(S)=0$), but still possible under the other model ($\nu(S)\neq 0$), which is exactly what equivalence of probability measures rules out.

If a price P(t) can be observed over an interval, say [0,1], we say it follows the stochastic process μ if μ is a probability measure over events defined as sets of possible observed time paths for P. Roughly speaking, P follows a process equivalent to a martingale process if there is a martingale process such that no possible observable behavior of P can allow us to tell with certainty whether we are observing the martingale process or P's actual probability distribution, and vice versa -- if we observed a Q generated by the martingale process there would be no possible observation that could rule out Q's having been generated by the P process.

If we can observe P only at discrete intervals, say t=j/n, j=1,...,n, equivalence to a martingale places no restrictions at all on P's observable behavior. The Wiener process martingale W(t), defined by its properties that W(t)-W(s) ~ N(0,t-s) and that changes in W over nonoverlapping time intervals are independent, gives W(j/n), j=1,...,n a joint normal distribution with nonsingular covariance matrix. This means of course that the vector of W values at the n points j/n has a p.d.f. that is continuous and nonzero over the whole of n-dimensional Euclidean space $\mathbb{R}^n$. But any distribution on $\mathbb{R}^n$ which has an everywhere nonzero p.d.f. is equivalent to Lebesgue measure, and hence all such distributions are mutually equivalent. Thus any stochastic process which gives P(j/n), j=1,...,n a nonzero p.d.f. over all of $\mathbb{R}^n$ is equivalent over those n discrete t-values to the Gaussian process generated by sampling a Wiener process at those values of t. This covers most of the stochastic processes used in practical time series modeling -- all ARIMA processes with Gaussian disturbances, for example.

When P can be observed at every point of the [0,1] interval, equivalence to a martingale is not an empty condition. For example, a Wiener process puts zero probability on the event that P(·) is differentiable over [0,1]. Some commonly used continuous time stochastic models (e.g., stochastic differential equations of higher than first order) imply that P(·) is differentiable with probability one. Thus it is easy to write down a convenient continuous time model that is not equivalent to a Wiener process.

However, the class of all continuous-time martingales is large, and it turns out that most commonly used, convenient models for continuous time stochastic processes are at least absolutely continuous with respect to <u>some</u> continuous time martingale, even though many are not absolutely continuous with respect to a Wiener process. That is, defining observable behavior for prices that has zero probability under <u>every</u> martingale process is not easy. Consider a process whose time paths have continuous derivatives with probability one, for example. We know such a process is not absolutely continuous with respect to a Wiener process, but it is absolutely continuous with respect to a different martingale.

In particular, suppose μ is the probability measure for the differentiable process and suppose that we generate a sequence of random times t_j, $j=1,\dots,\infty$, from a Poisson process that makes the probability of an event generating a new t_j .01 per unit time. (That is, at any date t, the p.d.f. of the time s until the next t_j is $.01e^{-.01s}$.) We define a new process as follows. To generate a random time path for the new process first draw a time path for P from the μ distribution, then a sequence of t_j's from the Poisson process. Modify the initially drawn P path by adding a discontinuous jump of height $-100\dot{P}(t_j)$ at each of the random times t_j. ($\dot{P}$ is the derivative of the original P with respect to t.) The resulting process will have differentiable time paths except for occasional random jumps, and will be a martingale. Given the way we have chosen the jump process, the odds are about 100 to 1 against observing any jump over the interval [0,1]. Thus the kind of behavior we always observe under μ -- differentiable time paths of P over [0,1] -- is usually observed over that interval under the martingale process we have constructed. The differentiable process is absolutely continuous with respect to the martingale we constructed from it, because if the differentiable process is the truth, we will never observe a path of P over [0,1] which allows us to be certain that the process is not the martingale we constructed. Of course, there is some probability under the martingale process that we will observe a discontinuous jump in P on [0,1], and in that case we could be sure we were not observing the differentiable process, so the martingale is not absolutely continuous with respect to the differentiable process.

2. An Observable Criterion for Failure of Martingale Equivalence

If the highly predictable behavior over short time intervals of a differentiable process is not ruled out by martingale equivalence, what kind of price behavior could we observe that would allow us to conclude that martingale equivalence fails? One kind of price behavior that would allow such a conclusion can be characterized as follows. For any process X that is a local martingale[2] there is an associated positive, increasing process $[X,X]_t$ called the (optional) *quadratic variation*[3] in X. We refer the reader to Jacod and Shiryayev [1987], §4e for a formal definition, as it requires some technical apparatus. However it can be understood through the following definition of an estimator of it and through a statement of some of its properties. We define

$$(2) \qquad [X \hat{,} X]_t^h = \sum_{j=1}^{[t/h]} (X(jh)-X((j-1)h))^2 \text{ , and then}$$

$$[X \hat{,} X]_t = \lim_{h \to 0} \left\{ [X \hat{,} X]_t^h \right\} \text{ ,}^4$$

where the notation "[t/h]" means "the largest integer less than or equal to t/h."

Jacod and Shiryayev [1987] (Theorem 4.47) shows that term in brackets on the right of (2) converges in probability as $h \to 0$ to $[X,X]_t$ when X is a local martingale. This does not imply that it converges for any single sample path, but it does imply that the probability of paths such that the limit on the right of (2) exists and is something other than $[X,X]_t$ is zero. A martingale has zero quadratic variation with probability one if and only if it is a trivial constant martingale. (See Jacod and Shiryayev [1987] Proposition 4.50.)

Let I_t be the set of all finite sequences

$$\left\{s_i\right\}_{i=1}^{n} \quad , \quad 0<s_i<s_{i+1}<t, \text{ all } i=1,...,n-1.$$

The total variation in X over [0,t] is then defined as

$$\text{(3)} \qquad V_X(t) = \sup_{\{s_j\} \text{ in } I_t} \left\{ \sum_{j=1}^{n} \left| X(s_j)-X(s_{j-1}) \right| \right\} .$$

Theorem: *If* X *is a local martingale, with probability one there is no interval* [0,t] *with* t>0 *such that* $[X,X]_t=0$ *and* $V_X(t)=\infty$.

Proof: See appendix.

Thus we have the desired criterion for an observed time path that could not have come from a martingale process: a path for which our estimate shows zero quadratic variation over some interval but that has unbounded variation (in the usual sense) over the interval.

A local martingale also cannot have infinite quadratic variation over finite intervals. This provides a second criterion for an observed time path that cannot have come from a martingale process.

Implementing a check for these condition with real data raises some problems. Both (2) and (3) include limit operations. In practice, we never have truly continuous records of asset prices, so we cannot actually compute these limits. We must be content with computing (2) for a finite sequence of n's which grow larger and (3) for a finite sequence of increasingly fine partitions. We would look for the finite sequence of right-hand sides of (2) to be shrinking while a finite sequence of right-hand sides of (3) is increasing. Of course this can never give us the certainty about failure of the no-arbitrage condition that would be possible with a truly continuous data record.

However, the fact that actual data cannot truly leave us certain that observed prices are not drawn from a martingale-equivalent process does not fatally weaken our results. The results carry the same kind of weight as a standard consistency proof. Just as, with an infinite sample, we can be certain that data are generated by an autoregression with a unit root, with a continuous record we can be certain that an observed price process is not martingale-equivalent. Just as our samples are in fact finite, they are in fact not truly continuous. Knowing what methods would give us certainty in an infinite sample (consistent estimation procedures) is a useful guide to good procedure in finite samples; similarly knowing what methods would give us certainty with a truly continuous record is a useful guide to good procedure with actual fine-time-unit, but discrete, data.

3. Convenient Parametric Models Not Absolutely Continuous with Respect to Martingales

Since our conclusions cannot be certain, we will want to place probabilistic error bounds on them. To do so, we will need to formulate a parametric model for the stochastic process followed by an observed asset price that includes both stochastic processes equivalent to martingales and stochastic processes that are not.

We first note that there is a well known class of processes some of which over finite intervals with probability one generate infinite quadratic variation and others of which with probability one generate zero quadratic variation and infinite total variation. What Mandelbrot and Van Ness [1968] call "fractional Brownian motions" have the property that when their parameter H satisfies $0<H<.5$, quadratic variation is infinite with probability one, and when $.5<H<1$, with probability one quadratic variation is 0 and total variation is infinite. Much of the literature on fractional Brownian motion emphasizes the "long dependence" properties of such processes -- the slow rate at which their autocovariance functions decay toward zero at infinity or the unboundedness of the spectral density or its derivative at 0. We focus instead on the processes' fine time unit properties -- the unboundedness of the derivative of the autocovariance function at 0 or the slow decay toward zero at infinity of the spectral density function.

Because the fractional Brownian motions tie the long dependence properties of the process to the same parameter that determines its fine time unit behavior, they do not form a practically useful parametric family for modeling asset prices. Building on work in S. Maheswaran [1990a,b], we show in the appendix that if X is a Gaussian process with moving average representation

(4) $$X(t) = \int a(t-s)dW(s) ,$$

if $a(s)/s^p$ converges to some $\varepsilon>0$ as $s\to 0$ from above, and if $a(s)$ is otherwise smooth, then for $-.5<p<0$, not only $[X,X]$ but our estimate $[\hat{X,X}]$ is infinite with probability one. Also, if $0<p<.5$, then with probability one $[X,X]$ and $[\hat{X,X}]$ are zero, while V_X is infinite. (Note that our p corresponds to H-.5 in the notation of Mandelbrot and Van Ness.)

This result opens up a range of practical modeling alternatives. For example, one could construct any convenient parametric family of functions $b(\cdot;\beta)$: $\mathbb{R}^+\to\mathbb{R}$ satisfying $b(0;\beta)=1$, b differentiable in its first argument t for all $t>0$ and right-differentiable at $t=0$, and $\int t^2 b(t;\beta)^2 dt<\infty$. Then letting a in (4) satisfy

(5) $$a(s) = \gamma s^p b(s;\beta)$$

one has a convenient parametric family of stationary, Gaussian processes allowing infinite quadratic variation ($-.5<p<0$), martingale-equivalent behavior ($p=0$), zero quadratic variation combined with infinite total (first-order) variation ($0<p\leq .5$), and zero quadratic variation combined with bounded total variation ($1\geq p>.5$).

For technical reasons it may be worth noting that processes of the class parameterized in (5) are, for $-.5<p<.5$, $p\neq 0$ not only not absolutely continuous with respect to any martingale measure, they are not semimartingales. That this is true follows directly from the definition of a semimartingale as the sum of a local martingale and a process whose paths have bounded total variation. Since much of the convenient apparatus of stochastic calculus fails to apply outside the realm of semimartingales, we should observe that we have here an example where assuming that an asset price follows a semimartingale is a substantive restriction, not a mere regularity condition.

4. Comparing Smoothness and Variation-Discrepancy as Criteria for Failure of the No-Arbitrage Condition

A price process P whose time paths are almost surely absolutely continuously differentiable in t is not equivalent to a martingale and offers a simply defined arbitrage opportunity.[5] Suppose we have a proposed rule for choosing over the interval $0 \le t \le 1$ a time path A(t) for holdings of the asset whose price is P(t) and also a path N(t) for holdings of the numeraire asset. Whatever the rule may be, we can improve on its performance with zero risk by changing the original N(t) path to $N(t)-\delta\dot{P}(t)$ for any $\delta>0$. To satisfy the original wealth constraint, the new path A*(t) for holdings of the non-numeraire asset will have to satisfy

$$\text{(6)} \qquad dA^*(t) = dA(t) + \delta\frac{d[\dot{P}(t)]}{P(t)} .$$

But it is easy to check, integrating by parts, that (6) implies

$$\text{(7)} \qquad P(t)A^*(t) + N(t) - \delta\dot{P}(t)$$
$$= P(t)A(t) + N(t) + \delta P(t)\int_0^t \left(\frac{\dot{P}(s)}{P(s)}\right)^2 ds.^6$$

The left-hand side of (7) is just the value of the portfolio under the new rule, and the right hand side shows that it strictly exceeds the value under the old rule. Harrison et al [1984] show that arbitrage opportunities exist for differentiable paths even when they are not absolutely continuously differentiable.

However, as we have already observed, this kind of deviation from the no-arbitrage condition cannot be detected with certainty from a finite span of data. As the period over which we observe differentiability lengthens, it would be reasonable for us to be more and more confident that the discontinuous jumps in P that would reconcile observed differentiability with martingale behavior will never occur. But no finite span of observation can make us certain of this, even with a continuous data record.

In contrast, the type of deviation from martingale equivalence we focus on in this paper, which we may as well call "variation discrepancy", can be identified with certainty from a finite span of data. It is true that checking for variation discrepancy requires observation at arbitrarily fine time units, but of course verifying differentiability also requires observation at arbitrarily fine time units.

The nature of arbitrage strategies that would exploit the failure of martingale equivalence via variation discrepancy is an interesting question. As we show in the appendix, for price processes whose logs are of the form (4), with a given by (5) with $-.5<p<.5$, $p\neq 0$, it is possible to construct trading rules based on the observed history of prices that obtain arbitrarily low ratios of standard deviation of return to mean return. The idea is that one trades at uniform time intervals h, always investing in the asset when the expected price change over the next interval of length h is positive, in the numeraire otherwise. The mean return from such a strategy over a fixed interval goes to infinity as the trading interval h goes to zero. The standard deviation of the return remains bounded.

As is also shown in the appendix, this possibility implies the existence of an arbitrage opportunity in the time-zero contingent claims market. The idea is that though none of these investment strategies is risk-free, as the risk becomes arbitrarily small their time-zero value as a random payout must eventually exceed the time-zero price of the capital necessary to implement them.

In the appendix we also construct purely risk-free arbitrage strategies entirely in the spot market. The idea is to choose h and the fraction of the portfolio invested in the asset to make the probability of loss over (0,.5) .25, say. Then if there is a gain by time t=.5, hold the numeraire until t=1. Otherwise, recalibrate h and the fraction of the portfolio invested so that the probability of failing to be above the initial t=0 level of wealth by t=.75 is less than .125. Repeat this indefinitely, at each date $t=1-2^{-j}$ stopping if a profit has been achieved, otherwise intensifying the transaction rate so that the probability of not achieving a profit over the next interval of length 2^{-j-1} is less than 2^{-j-2}. This achieves a profit over the interval (0,1) with probability one, using only a fixed initial investment.

5. Conclusion

We have shown that there are usable probability models for price behavior that are outside the semimartingale class and that may have a role in financial market modeling when absence of arbitrage opportunities is not a foregone conclusion. We hope to have encouraged the interest of other researchers who may help to discover the ultimate usefulness of these ideas.

Appendix

A. A Measure-Theoretic Derivation of Martingale Equivalence

We consider two assets, one the numeraire and the other, which we call c, whose spot price in terms of the numeraire is given by P_t for t in [0,1]. We assume that there are some traders in the market who hold both assets and who care about the total value of their portfolio at t=1, but not about their asset holdings at intermediate points in time.[7] There is a probability space Ω and a sigma-field F of subsets of Ω. The information structure, common across all agents, is defined by the increasing family of sigma-fields $\{F_t$: t in $[0,1]\}$ with $F_1=F$. We suppose that trading is allowed in arbitrary contingent claims on the numeraire asset and c, subject to the requirement that an asset traded at t can be contingent only on events in F_t.

Let δ^c,δ^n: $F \to \mathbb{R}^+$ denote set functions such that $\delta^n(A)$ and $\delta^c(A)$ are the prices at time zero of one unit of the numeraire asset or c, respectively, delivered at t=1 contingent on the true state of the world ω lying in A. Since one unit of either asset can be delivered regardless of the state of the world at t=1 by simply holding it from t=0 to t=1, we must have $\delta^n(\Omega)=1$, $\delta^c(\Omega)=P_0$. We also assume $\delta^c,\delta^n \geq 0$ for all arguments, as usual for a price. We take the absence of arbitrage opportunities to imply that each δ be a measure. That is, it must be true that $\delta(\emptyset)=0$ and that for any countable class $\{A_j\}$ of disjoint sets in F

$$\delta\left(\cup A_j\right) = \Sigma\, \delta\left(A_j\right) . \tag{A1}$$

If (A1) did not hold, there would be some class of claims, contingent on a set of mutually exclusive events, that could be bought separately at a total price different from the price of an equivalent claim contingent on the union of the separate contingencies. In other words, two equivalent bundles of goods would have different prices, which is exactly an arbitrage opportunity.[8]

Now suppose that there is no set A in *F* such that $\delta^n(A)=0$ and $\delta^c(A)>0$. For this condition to fail would imply that there is some contingency under which the numeraire is valueless but the other asset is still valuable. This condition means that δ^c is absolutely continuous with respect to δ^n. It is a standard result in measure theory that under these conditions there is an integrable function (the Radon-Nikodym derivative) $Q:\Omega\to\mathbb{R}$ satisfying

(A2) $$\text{for every A in } F,\ \delta^c(A) = \int_A Q(\omega)\ \delta^n(d\omega)\ .$$

Now it is well known that for any sub-σ-field F_t of F, there is an integrable, F_t-measurable function $E_t[Q|\cdot]$ on Ω with the property that

(A3) $$\text{for every A in } F_t,\ \delta^c(A) = \int_A E_t[Q|\omega]\ \delta^n(d\omega)\ .$$

We will proceed to show that with probability one $P_t=E_t[Q]$.

To do so we have to extend our δ's to pricing rules for a wider class of payouts than the simple unit contingent payouts for which they are already defined. It is natural to suppose that countable linear combinations of contingent payouts can be priced, with a payout of $\lambda_i>0$ units of the numeraire for ω in A_i, priced at

$$\text{(A4)} \qquad \pi\left(\Sigma\ \lambda_i\ I(\omega;A_i)\right) = \Sigma\ \lambda_i\delta^n(A_i)\ .$$

Here we have represented the payout as a linear combination of indicator functions on Ω, with π a "pricing function" delivering current prices of payouts specified as functions on Ω. That this type of payout be priced this way is also a consequence of a no-arbitrage assumption, assuming that contingent claims are infinitely divisible. From (A4) π already has the form of an integral, and we can extend π uniquely to all positive δ^n-measurable functions on Ω by using one further property of arbitrage: if a payout f on Ω dominates another payout g in the sense that $f(\omega)>g(w)$ for all ω except for a set of δ^n-measure 0, then $\pi(f)\geq\pi(g)$. Given this, it must be true that

$$\text{(A5)} \qquad \pi(f) = \int f(\omega)\ \delta^n(d\omega)$$

for any F-measurable f[9] .

Consider an arbitrary set A in F_t. We consider how to value a claim to a unit of c contingent on A. We have two feasible ways of delivering one unit of c at t=1 contingent on A: Buy at t=0 a claim to one unit of c contingent on A, or buy at t=0 an asset that pays P_t units of the numeraire good at t, contingent on A being realized at t. The proceeds from this asset can be used at t to purchase a unit of c if A has been realized. We know how to price these two assets. The outright contingent claim to c is worth at t=0, by (A2)

(A6) $$\delta^c(A) = \int_A E_t[Q \mid \omega] \; \delta^n(d\omega)$$

while the claim to the P_t payout contingent on A is worth, by (A5),

(A7) $$\pi(P_t I(\omega;A)) = \int_A P_t(\omega) \; \delta^n(d\omega) \; .$$

Since absence of arbitrage opportunities implies (A6)=(A7) for all A in F_t, except on a set of δ^n-measure one $P_t=E_t[Q]$.

Observe that so far we have not made any use of the notion of a true or "physical" probability distribution on Ω. To do so, we need to add an assumption about the relation between δ^n-measure and true probability. It is natural to suppose they are equivalent, so that no value is given to delivery of a unit of the numeraire contingent on an event of zero probability, and some positive value is given to delivery of a dollar contingent on any event that has non-zero probability. Then the P_t stochastic process, which we have shown to be a martingale under the δ^n measure, is equivalent to a martingale under the true probability measure.

B. Properties of Processes with MA Kernel Behaving like s^p Near the Origin

Consider a process X of the form (4) with a of the form (5), satisfying

i) $b(0)\neq 0$;

ii) b(s) right-continuous in s at s=0;

iii) $b'(s)$ exists and is bounded and continuous;

iv) $-.5<p<.5$.

(Here we have suppressed the argument β in (5).) It is convenient to define

$$\Delta_h X(t) = X(t)-X(t-h) \quad .$$

Lemma 1: *For an* X *satisfying (i)-(iv), for every integer* j

(A8) $$R_h(j) = h^{-(2p+1)}\mathrm{Cov}\Big(\Delta X_h(t),\Delta X_h(t-jh)\Big) \xrightarrow[h\to 0]{} R(j) \ ;$$[10]

and there exists a C>0, *not dependent on* h, *such that for all* j>0,

(A9) $$Cj^{p-1} > |R_h(j)| \quad \textit{for } -.5<p<0,$$
$$Cj^{2p-1} > |R_h(j)| \quad \textit{for } 0<p<.5 \ .$$

Proof:

$\Delta_h X(t)$ has moving average kernel $\alpha_h(s)$, with $\alpha_h(s)=a(s)$, $0\le s<h$, $\alpha_h(s)=a(s)-a(s-h)$ for $s\ge h$. Thus we can write for $j\ge 1$

$$h^{2p+1}R_h(j) = \int_0^h b(s)s^p\Big((s+jh)^p b(s+jh)-(s+jh-h)^p b(s+jh-1)\Big)ds$$

$$\text{(A10)} \qquad + \int_h^\infty \Big(b(s)s^p-b(s-h)(s-h)^p\Big)\Big((s+jh)^p b(s+jh)-(s+jh-h)^p b(s+jh-h)\Big)ds \ .$$

For j=0, the first term in (A10) is replaced by $\int_0^h b(s)^2 s^p ds$.

Suppose in (A10) or the corresponding expression for j=0 we make the change in variables s=vh. Then it is straightforward to check that, by the boundedness of b and its derivative, the two integrals in (A10) behave like h^{2p+1} as $h \to 0$ for any fixed j. (Checking this does require noting that $v^p-(v-1)^p$ behaves like v^{p-1} as $v \to \infty$, and is thus integrable for p<.5.) Further, there is a constant C>0 such that the first integral on the right-hand side of (A10) is bounded above in absolute value by

$$\text{(A11)} \qquad Ch^{2p+1}\int_0^1 v^p\Big|(v+j)^p-(v+j-1)^p\Big|dv < Ch^{2p+1}j^{p-1}$$

for j≥2, where the generic "C" may be different on the two sides of (A11). This follows by noting that, using a Taylor expansion of the second factor in the middle expression of (A12) below,

$$\text{(A12)} \qquad \Big|(v+j)^p-(v+j-1)^p\Big| = j^p\left|\left(\frac{v}{j}+1\right)^p-\left(\frac{v-1}{j}+1\right)^p\right| \cong j^{p-1}$$

With the same change of variable we can bound the second integral on the right-hand side of (A10) by

$$Ch^{2p+1}\int_1^{\infty} v^{p-1}(v+j)^{p-1}\,dv = Ch^{2p+1}\int_1^{\infty}(v^2+vj)^{p-1}dv$$

$$\text{(A13)} \qquad \leq Ch^{2p+1}j^{2p-1} .$$

Here as before we use a generic constant "C" that may take on different values in different expressions. Note that when p<0 (A11) becomes the effective bound on (A10), while when p>0 (A13) is the effective bound. □

Lemma 2: *For* p>0, $[\hat{X,X}]_1^h \xrightarrow{a.s.} 0$ *as* $h \to 0$.

Proof:

From (A8) it is easy to conclude that

$$\text{(A14)} \qquad E\left[[\hat{X,X}]_1^h\right] = E\left[\sum_{j=1}^{h^{-1}} \Delta_h X(jh)^2\right] \text{ behaves like } h^{2p} \text{ as } h \to 0 .$$

Since $[\hat{X,X}]_1>0$ with probability one, the fact that h^{2p} in (A14) goes to zero with h for p>0 implies that for this case $[\hat{X,X}]_1^h$ converges in probability to zero as $h \to 0$. By choosing a sequence h_j that converges fast enough to zero ($h_j=\delta^j$ for any $0<\delta<1$, for example) we can make the convergence almost sure.[11]

Lemma 3: *For* p<0, $[\hat{X,X}]_1^h \xrightarrow{a.s.} \infty$ *as* $h \to 0$.

Proof:

First we need

Lemma 4:

For X *and* Z *jointly Gaussian,*

(A15)
$$\mathrm{Cor}(X^2,Z^2) = \Big(\mathrm{Cor}(X,Z)\Big)^2 .$$

In (A15) "Cor" means "correlation".

For $p<0$, (A9) implies that R_h is absolutely summable. Then (A8)-(A10) can be used, together with (A15), to show

(A16)
$$h^{-4p-1}\mathrm{Var}\Big(\widehat{[X,X]}_1^h\Big) \xrightarrow[h\to 0]{} \varepsilon \quad .$$

Thus the ratio of the mean of $[X,X]_1^h$ to its standard deviation behaves like $h^{-.5}$ and goes to infinity as $h\to 0$. This guarantees that $[X,X]_1^h$ converges in probability to infinity as $h\to 0$ for $p<0$. Again, by choosing a sequence of h's converging rapidly enough to zero, we can attain convergence with probability one. ▫

It remains then to show

Lemma 5: *The paths of* X *will be of unbounded variation when* $0<p<.5$.

Proof:

Observe that

(A17) $$V_X^h(1) = \sum_{j=1}^{h^{-1}} \left|\Delta_h X(j/h)\right| .$$

The individual random variables in the sum on the right-hand side of (A9) have expected value bounded below by a term of the form $Ch^{p+.5}$, $C>0$, so the entire term has expectation bounded below by a $Ch^{p-.5}$ term that goes to infinity as $h \to 0$.

For jointly Gaussian random variables Y and Z with correlation ρ,

(A18) $$\mathrm{Cor}\left(|Y|,|Z|\right) \text{ behaves like } \rho^2 \text{ as } \rho \to 0 .$$

This, together with (A9), allows us to conclude that the standard deviation of $V_x^h(1)$ behaves like h^{1-2p} for small h and thus again to conclude that we have convergence in probability to infinity, with convergence almost surely if the h sequence is taken to converge quickly enough to zero. □

C. Arbitrage Strategies

Since non-martingale-equivalent price processes must present arbitrage opportunities, it is interesting to explore what these might be. For spot prices Q_t whose logarithms q_t follow processes of the type considered in Appendix C above, we can construct arbitrage strategies as follows. First we observe the properties of the following type of investment: At date 0, invest \$1 (one unit of the numeraire) in the asset if $E_0[\Delta_h q_h]>0$, hold on to the \$1 (invest it in the numeraire) otherwise. Continue to follow this rule, investing the entire current accumulation of wealth in the asset when $E_t[\Delta_h q_{t+h}]>0$, keeping it in the numeraire otherwise, with the portfolio being adjusted according to this rule at each date t that is an exact integral multiple of h. Over the fixed interval $[0,\gamma]$, if W_t is the value of the portfolio at date t, we have

$$\text{(A19)} \qquad E_0\left[\log\ W_\gamma\right] = \frac{\gamma}{2h} E_0\left[E_t\left[q_{t+h}-q_t\right] \;\middle|\; E_t\left[q_{t+h}-q_t\right] > 0\right].$$ [12]

Now $E_t[q_{t+h}-q_t]$ is the "explained part" of $\Delta_h q_{t+h}$, based on all information in the continuous record of q up to time t. It therefore has larger variance than the best predictor of $\Delta_h q_{t+h}$ based on $\Delta_h q_t$ alone. By (A8) we know that the latter variance behaves like h^{2p+1}, and is thus bounded below, for $p\neq 0$ and small h, by Bh^{2p+1} for some $B>0$. Assuming q is Gaussian, (A19) is just the expectation of a truncated Gaussian variable and is therefore proportional to the standard deviation of the variable. From the form of (A19), we can see that it is bounded below by $B_1h^{p-.5}$ for some $B_1>0$, so long as $p\neq 0$. With $-.5<p<.5$, we can thus conclude that expected return over the interval $[0,\gamma]$ is arbitrarily large for h small enough.

To determine the variance of the return we denote the return over one h-length interval as

$$\text{(A20)} \qquad y_{t+h} = \Delta q_{t+h} \cdot I\left(E_t\left[\Delta q_{t+h}\right] > 0 \right),$$

where $I(\cdot)$ is a random variable taking on the value one when its argument is true and zero otherwise. Then the autocovariance function of y satisfies, for $|i-j|\geq 1$,

(A21)
$$\left|\mathrm{Cov}\left(y_{ih},y_{jh}\right)\right| \le \begin{Bmatrix} K\,|i-j|^{\,p-1}h^{2p+1}, & -.5<p<0 \\ K\,|i-j|^{2p-1}h^{2p+1}, & 0<p<.5 \end{Bmatrix} .$$

This set of inequalities follows from (A9) and the fact that for any two jointly Gaussian random variables X and Y with correlation coefficient ρ and any functions f and g such that $|f(X)|$ and $|g(Y)|$ have finite first and second moments, the correlation of f(X) and g(Y) is $0(\rho)$ as $\rho\to 0$. The variance of W_γ is the variance of the sum of the y_{ih}'s over the $(0,\gamma)$ interval and is thus bounded by

(A22)
$$\frac{\gamma}{h}\sum_{j=-\gamma/h}^{\gamma/h} Kj^{\pi-1}h^{2p+1} ,$$

where $\pi=p$ for $p<0$, $\pi=2p$ for $p>0$. This is in turn bounded above by

(A23)
$$\frac{\gamma}{h}\,K\,\left(\frac{\gamma}{h}\right)^{\pi}h^{2p+1} = \begin{Bmatrix} 0(h^{p}), & p<0 \\ 0(1), & p>0 \end{Bmatrix} ,$$

where the value of the constant K may shift between its occurrences in (A22) and (A23). Since we found above that the expected return is bounded below by $B_1h^{p-.5}$, we can apply (A23) to find that the ratio of expected return to its standard deviation goes to infinity as $h\to 0$, behaving as $h^{p-.5}$ for $0<p<.5$ and as $h^{.5(p-1)}$ for $-.5<p<0$.

The strategies we have outlined here do not directly produce a risk-free gain, of course, since there is always some non-zero variance to the return they provide. They can be used to develop strictly risk-free arbitrage opportunities in two ways.

First, in the presence of contingent claims markets, it must be true that as risk goes to zero while expected return remains bounded below, the value of contingent claims to the payout must converge to a number greater than the initial capital required. Thus there will be an arbitrage opportunity in the contingent claims market.

To produce an arbitrage opportunity in the spot market, we apply a strategy that looks like the classic doubling strategies for martingale processes. Apply the strategy outlined above over the time interval (0,.5), with h and the fraction of the initial portfolio committed to the investment strategy in this initial period chosen so that the probability of losing any money over the interval is less than .25. If at t=.5 we have made a profit, we stop and hold our winnings until t=1. Otherwise, we readjust h and the fraction of our portfolio invested so that the probability of our wealth being below the initial t=0 level at t=.75 is less than .125. That is, we choose h so that the expected gain over (.5,.75) is high enough, and the variance low enough, that we will be above the initial wealth level at t=.75 with probability .875. Now we repeat this indefinitely, at each time $1\text{-}2^{-j}$ either stopping, because we have achieved a profit, or intensifying the rate of trading so as to increase the expected rate of return and make the probability of failing to achieve a profit by the end of the next interval of length 2^{-j-1} less than 2^{-j-2}. It is easily seen that the probability of never achieving a profit during the time interval (0,1) is zero. Further, the total number of transactions required to implement the strategy will be finite, though there is no deterministic upper bound on the number of transactions that will be required.

C. Proof of the Theorem

First we need another result.

Lemma 6: *A purely discontinuous local martingale* X *that has a.s. no more than a single point of discontinuity has locally absolutely integrable variation.*

Remark: This lemma may appear more obvious than it is. A martingale with a randomly timed jump consists in general of two components, the jump part and a "compensator" with continuous sample paths. The work in the proof is to show that the compensator, besides being continuous, necessarily has finite variation.

Proof: Let T be the stopping time defined as the date at which X jumps. Then the "sum of jumps" process for X is

(A24)
$$\sum_{s\le t} \Delta X = \begin{cases} 0, & t<T \\ X(T)-X^{-}(T), & t\ge T \end{cases} .$$

This process is obviously a.s. of finite variation. It is also locally of absolutely integrable total variation. To see this, observe that we can define the sequence of stopping times

(A25)
$$T_n = \lim\ \inf \left\{ t:\ |X(t)|>n \right\}$$

so that $T_n \to \infty$ a.s. and the stopped martingales X^{T_n} are all absolutely integrable. To simplify notation we denote X^{T_n} by X_n. We have

(A26)
$$X_n(T) = \left(X_n(T)-X_n^{-}(T) \right) + X_n^{-}(T) \quad .$$

$X_n(T)$ and $X_n^-(T)$ are by construction absolutely integrable, which implies that the term in brackets must be also. Thus the process

(A27) $$\sum_{s\leq t} \Delta X_n = \left(\sum_{s\leq t} \Delta X\right)^{T_n}$$

is absolutely integrable and the process $\sum\Delta X$ itself is locally absolutely integrable (i.e., in $\mathcal{A}_{loc}$). But then, by Jacod and Shiryaev [1987] Theorem 3.18, there is a predictable process

(A28) $$\left(\sum_{s\leq t} \Delta X\right)^p \text{ in } \mathcal{A}_{loc} \text{ ,}$$

unique up to an evanescent set, such that

(A29) $$\left(\sum_{s\leq t} \Delta X\right) - \left(\sum_{s\leq t} \Delta X\right)^p$$

is a local martingale. By the uniqueness of (A28), the process displayed in (A29) must be X itself (up to an evanescent set). Otherwise, the difference between X and (A29) would be a non-zero local martingale with no discontinuities, and the assumption of pure discontinuity for X implies it contains no such component. Since both terms in (A29) have been verified to lie in A_{loc}, their difference, X, is in A_{loc} also. □ (lemma)

Now consider the result we are aiming for, restated here.

Theorem: *If* X *is a local martingale, with probability one there is no interval* [0,t] *with* t>0 *such that* $[X,X]_t=0$ *and* $V_X(t)=\infty$.

Proof: Consider the stopping time T = lim inf {t: $[X,X]_t>0$}. The stopped process X^T, defined as $X^T(t)=X(t\wedge T)$ (with $t\wedge T=\min(t,T)$), is a local martingale with at most one discontinuity -- a possible jump at T. Therefore by the lemma it has bounded variation with probability one. Since [0,T] is the longest interval on which [X,X] remains zero, we have proved the result. □

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[1]Our approach uses a slightly different set of primitives than the original Harrison-Kreps approach. By defining an arbitrage opportunity more broadly, we are able to use measure-theoretic structure more heavily and avoid explicitly introducing a topology on payoff functions.

[2]The word "local" here is used in the special sense of continuous time stochastic process theory. We do not attempt to define it here, but readers unfamiliar with it can note that a martingale is also locally a martingale and can refer to Jacod and Shiryayev [1987] for more detail.

[3]We will henceforth simply call this the quadratic variation.

[4]Note that this estimate is a random variable chosen to approximate another random variable -- $[X,X]_t$ -- and not a conventional statistical estimator of a nonrandom parameter.

[5]This assertion might seem to contradict the earlier claim that we can never be sure that a smooth price process is not drawn from a martingale-equivalent process. The arbitrage opportunity demonstrated here depends on it being *certain* that the price process is differentiable.

[6]From (6) we have

$$A^*(t)-A^*(0) = A(t)-A(0)+\delta\int_0^t \frac{\ddot{P}(s)}{P(s)}\,ds$$

$$= A(t)+\delta\left|_0^t \frac{\dot{P}(s)}{P(s)} + \int_0^t \left(\frac{\dot{P}(s)}{P(s)}\right)^2 ds\right.$$

This yields (7) when multiplied by P(t).

[7]This rules out assets that have dividend or interest payments or that provide a utility yield as they are held. The martingale equivalence result proved below can be extended to assets that do have such yields. Yields must occur in a continuous flow rather than as discrete payouts at isolated dates, but otherwise the extension puts little restriction on the nature of yields. Such an extension is displayed in Sims [1984].

[8]We should observe, though, that it is probably more conventional to require only that <u>finite</u> collections of contingent claims sell at the same price as a claim contingent on the union of the contingencies they represent. Countable additivity is then derived from a separate assumption of continuity in some topology on the space of return functions.

[9]Note that $\pi(f)$ may be infinite if f is unbounded.

[10]We will sometimes below state conditions like (A10) in the form "f(h) behaves like g(h) as $h\to 0$", meaning "$f(h)/g(h)\to\varepsilon$ as $h\to 0$ for some $\varepsilon\neq 0$".

[11] $E[|X_j|]=0(\delta(j))$, where $\delta(j)$ decreases monotonically to 0 as $j\to\infty$, implies $P[|X_j|>\varepsilon]=0(\delta(j))$. By choosing k(j) to increase rapidly enough in j, e.g. so that $k(j)=\delta^{-1}(2^{-j})$, we can make $P[|X_{k(j)}|>\varepsilon]=0(2^{-j})$, and thus guarantee that

$$\sum_{i=j}^{\infty} P[|X_{k(j)}|>\varepsilon] \underset{i\to\infty}{\longrightarrow} 0 \text{ , i.e. } X_{k(j)} \xrightarrow{a.s.} 0 \text{ .}$$

[12] Since $\log(W_0)=0$, $\log(W_\gamma)$ is the sum of the changes in logs of W over the length-h decision intervals. The unconditional probability that $E_t[q_{t+h}-q_t]>0$ is .5, and the expected change in the log portfolio value during periods when it is kept entirely invested in the numeraire is 0. Thus the right-hand-side of (A16) is just the unconditional expectation of the change in log portfolio value over a typical interval, multiplied by the number of intervals. This argument does depend on E_0 being the same as unconditional expectation, which requires that the process "start up" at time 0 with zero innovation process before that.