SOLVING LINEAR RATIONAL EXPECTATIONS MODELS

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1. General form of the models

The models we are interested in can be cast in the form

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi \eta(t) \tag{1}$$

$t = 1, \ldots, T$, where C is a vector of constants, $z(t)$ is an exogenously evolving, possibly serially correlated, random disturbance, and $\eta(t)$ is an expectational error, satisfying $E_t \eta(t+1) = 0$, all t. The $\eta(t)$ terms are not given exogenously, but instead are treated as determined as part of the model solution. Models with more lags, or with lagged expectations, or with expectations of more distant future values, can be accommodated in this framework by expanding the y vector. This paper's analysis is similar to that of Blanchard and Kahn (1980) with four important differences:

(i) They assume regularity conditions as they proceed that leave some models encountered in practice outside the range of their analysis, while this paper covers all linear models with expectational error terms.

(ii) They require that the analyst specify which elements of the y vector are predetermined, while this paper recognizes that the structure of the Γ_0, Γ_1, Ψ, and Π matrices fixes the list of predetermined variables. Our approach therefore handles automatically situations where linear combinations of variables, not individual variables, are predetermined.

(iii) This paper makes an explicit extension to continuous time, which raises some distinct analytic difficulties.

(iv) They assume that boundary conditions at infinity are given in the form of a maximal rate of growth for any element of the *y* vector, whereas this paper recognizes that in general only certain linear combinations of variables are required to grow at

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bounded rates and that different linear combinations may have different growth rate restrictions.

There are other more recent papers dealing with models like these (King and Watson, 1997, 1998; Anderson, 1997; Klein, 1997) that, like this one, expand the range of models covered beyond what is covered in Blanchard and Kahn's paper, particularly to include singular Γ_0 cases. All these papers, though, follow Blanchard and Kahn in requiring the specification of "jump" and "predetermined" *variables*, rather than recognizing that in equilibrium models expectational residuals more naturally are attached to equations. Also, only Anderson (1997) has discussed the continuous time case.

Less fundamentally, this paper uses a notation in which time arguments or subscripts relate consistently to the information structure: variables dated t are always known at t. Blanchard and Kahn's use of a different convention often leads to confusion in the application of their method to complex models.

An instructive example to illustrate how we get a model into the form (1) is a model of overlapping contracts in wage setting along the lines laid out by Taylor.

$$\begin{aligned} w(t) &= \tfrac{1}{3}E_t\left[W(t)+W(t+1)+W(t+2)\right]-\alpha(u(t)-u_n)+\nu(t) \\ W(t) &= \tfrac{1}{3}\left(w(t)+w(t-1)+w(t-2)\right) \\ u(t) &= \theta u(t-1)+\gamma W(t)+\mu+\varepsilon(t)\ , \end{aligned} \tag{2}$$

where $E_t\nu(t+1)=E_t\varepsilon(t+1)=0$. To cast (2) into the form (1) requires using the expanded state vector

$$y(t)=\begin{bmatrix} w(t) \\ w(t-1) \\ W(t) \\ u(t) \\ E_tW(t+1) \end{bmatrix} \tag{3}$$

With this definition of y, (2) can be written in the matrix notation of (1), with definitional equations added, by defining

$$\Gamma_0 = \begin{bmatrix} 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & -\gamma & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \alpha & 0 & \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \theta & 0 \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & \end{bmatrix},$$

$$C = \begin{bmatrix} \alpha \cdot u_n \\ 0 \\ \mu \\ 0 \\ 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{4}$$

$$z(t) = \begin{bmatrix} \varepsilon(t) \\ \nu(t-1) \end{bmatrix}.$$

This example illustrates the principle that we can always get a linear model into the form (1) by replacing terms of the form $E_t x(t+1)$ with $y(t) = E_t x(t+1)$ and adding to the system an equation reading $x(t) = y(t-1) + \eta(t)$. When terms of the form $E_t x(t+s)$ appear, we simply make a sequence of such variable and equation creations. It is often possible to reach the form (1) with fewer new variables by replacing expectations of variables in equations with actual values of the same variables, while adding $\eta(t)$ disturbance terms to the equation. While this approach always produces valid equations, it may lose information contained in the original system and thereby imply spurious indeterminacy. The danger arises only in situations where a single equation involves some variables of the form $E_t X(t+1)$ and other variables of the form $Y(t+1)$. In this case, dropping the E_t operators from the equation and adding an $\eta(t)$ error term loses the information that some variables are entering as actual values and others as expectations.

The formulation that most writers on this subject have used, following Blanchard and Kahn, is

$$\Gamma_0 E_t y(t+1) = \Gamma_1 y(t) + C + \Psi z(t). \tag{5}$$

One then adds conditions that certain variables in the y vector are “predetermined", meaning that for them $E_t y(t+1) = y(t+1)$. The formulation adopted here embodies the useful notational convention, that all variables dated t are observable at t; thus no separate list of what is predetermined is needed to augment the information that can be read off from the equations themselves. It also allows straightforward handling of the commonly occurring systems in which a variable $y_i(t)$ and its expectation $E_{t-1} y_i(t)$ both appear, with the same t argument, in the same or different equations. To get such systems into the Blanchard-Kahn form requires introducing an extended state vector. For example, in our simple overlapping contract model (2) above, the fact that W appears as an expected value in the first equation and/or in the dummy equation defining the artificial state variable $\zeta(t) = E_t W(t+1)$ (the bottom row of (4)), yet also in two other dynamic equations without any expectation, seems to require that, to cast the model into standard Blanchard-Quah form, we add $w(t-2)$ and $u(t-1)$ to the list of variables, with two corresponding additional dummy equations.

Standard difference equations of the form (1) have a single, exogenous disturbance vector. That is, they have $\Gamma_0 = I$ and $\Pi = 0$. They can therefore be interpreted as determining $y(t)$ for $t > t_0$ from given initial conditions $y(t_0)$ and random draws for $z(t)$. In (1), however, the disturbance vector $\Psi z(t) + \Pi\eta(t)$ defined in (1) is not exogenously given the way $z(t)$ itself is. Instead, it depends on $y(t)$ and its expectation, both of which are generally unknown before we solve the model. Because we need to determine $\eta(t)$ from z(t) as we solve the model, we generally need to find a number of additional equations or restrictions equal to the rank of the matrix in order to obtain a solution.

2. Canonical Forms and Matrix Decompositions

Solving a system like (1) subject to restrictions on the rates of growth of components of its solutions requires breaking its solutions into components with distinct rates of growth. This is best done with some version of an eigenvalue-eigenvector decomposition. We are already imposing a somewhat stringent canonical form on the equation system by insisting that it involve just one lag and just one-step-ahead expectations. Of course as we made clear in the example above, systems with more lags or with multi-step and lagged expectations can be transformed into systems of the form given here, but there may be some computational work in making the transformation. Further tradeoffs between system simplicity and simplicity of the solution process are possible.

To illustrate this point, we begin by assuming a very stringent canonical form for the system: $\Gamma_0 = I$, $E_t z(t+1) = 0$, all t. Systems derived from even moderately large rational expectation equilibrium models often have singular Γ_0 matrices, so that simply "multiplying through by Γ_0^{-1}" to achieve this canonical form is not possible. In most economic models, there is little guidance available from theory in specifying properties for z. The requirement that $E_t z(t+1) = 0$ is therefore extremely restrictive.

On the other hand, it is usually possible, by solving for some variables in terms of others and thereby reducing system size, to manipulate the system into a form with non-singular Γ_0. And it is common practice to make a model tractable by assuming a simple flexible parametric form for the process generating exogenous variables, so that the exogenous variables themselves are incorporated into the y vector, while the serially uncorrelated z vector in the canonical form is just the disturbance vector in the process generating the exogenous variables. So this initial very simple canonical form is in some sense not restrictive.

Nonetheless, getting the system into a form with non-singular Γ_0 may involve very substantial computational work if it is done ad hoc, on an even moderately large system. And it is often useful to display the dependence of y on current, past, and expected future exogenous variables directly, rather than to make precise assumptions on how expected future z's depend on current and past z's. For these reasons, we will below display solution methods that work on more general canonical forms. While these more general solution methods are themselves harder to understand, they shift the burden of analysis from the individual economist/model-solver toward the computer, and are therefore useful.

3. Using the Jordan decomposition with serially uncorrelated shocks

3.1. **Discrete time.** In this section we consider the special case of

$$y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi\eta(t)\,, \tag{6}$$

with $E_t z(t+1) = 0$. The system matrix Γ_1 has a Jordan decomposition

$$\Gamma_1 = P\Lambda P^{-1}\,, \tag{7}$$

where P is the matrix of right-eigenvectors of Γ_1, P^{-1} is the matrix of left-eigenvectors, and Λ has the eigenvalues of Γ_1 on its main diagonal and 0's everywhere else, except that it may have $\lambda_{i,i+1} = 1$ in positions where the corresponding diagonal elements satisfy $\lambda_{ii} = \lambda_{i+1,i+1}$. Multiplying the system on the left by P^{-1}, and defining $w = P^{-1}y$, we arrive at

$$w(t) = \Lambda w(t-1) + P^{-1}C + P^{-1}\cdot(\Psi z(t) + \Pi\eta(t))\,. \tag{8}$$

In this setting, we can easily consider non-homogeneous growth rate restrictions. That is, suppose that we believe that a set of linear combinations of variables, $\phi_i y(t)$, $i = 1,\dots,m$, have bounded growth rates, with possibly different bounding growth rates for each i. That is we believe that a solution must satisfy

$$E_s\left[\phi_i y(t)\xi_i^{-t}\right] \underset{t\to\infty}{\longrightarrow} 0 \tag{9}$$

for each i and s, with $\xi_i > 1$ for every i. Equation (8) has isolated components of the system that grow at distinct exponential rates. The matrix has a block- diagonal structure, so that the system breaks into unrelated components, with a typical block having the form

$$w_j(t) = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda_j & 1 \\ 0 & \cdots & 0 & 0 & \lambda_j \end{bmatrix} w_j(t-1) + P^{j\cdot}C + P^{j\cdot}\cdot(\Psi z(t) + \Pi\eta(t))\,, \tag{10}$$

where $P^{j\cdot}$ is the block of rows of P^{-1} corresponding to the j'th diagonal block of (8). If the disturbance term (including the combined effects of z and η on the equation) is zero and $\lambda_j \neq 1$, this block has the deterministic steady-state solution

$$w_j(t) = \left[I - \Lambda_j\right]^{-1} P^{j\cdot} C\,, \tag{11}$$

where Λ_j is the Jordan block displayed in (10). If $\left|\lambda_j\right| > 1$, then $E_s\left[w_j\left(t+s\right)\right]$ grows in absolute value at the rate of $\left|\lambda_j\right|^t$ as $t \to \infty$, for any solution other than that given in (11). Now consider the k'th restriction on growth,

$$\begin{aligned} E_s\left[\phi_k y(t)\right]\xi_k^{-t} &= \phi_k P E_s\left[w(t)\right]\xi_k^{-t} \\ &= \phi_k P\left[\Lambda^{t-s}(w(s) - (I-\Lambda)^{-1}P^{-1}C) + (I-\Lambda)^{-1}P^{-1}C\right]\xi_k^{-t} \to 0\,. \end{aligned} \tag{12}$$

In order for this condition to hold, every one of the vectors w_j corresponding to a $\left|\lambda_j\right| > \xi_k$ and to a $\phi_k P_{\cdot j} \neq 0$ must satisfy (11). This is obvious for cases where Λ_j is scalar. When Λ_j is a matrix with ones on the first diagonal above the main diagonal, a somewhat more elaborate argument is required. We need to observe that we can expand terms of the form on the right of (12) as

$$x\Lambda_j^s q = \lambda_j^s x \cdot \left(q + s\lambda_j^{-1}q_{-1} + \frac{s\cdot(s-1)}{2}\lambda_j^{-2}q_{-2} + \ldots + c_n(s)\lambda_j^{-n+1}q_{-n+1}\right)\,, \tag{13}$$

where $c_k(s)$ is the (k,s)'th binomial coefficient, i.e. $s!/((s-k)!\cdot k!)$, for $s \leqslant k$, and is 0 otherwise, and q_{-k} is the vector q shifted up by k with the bottom elements filled out with zeros, i.e.

$$q_{-k} = \left[\begin{array}{c|c} \underset{(n-1)\times 1}{0} & I \\ \hline \underset{1\times 1}{0} & \underset{1\times(n-1)}{0} \end{array}\right] \cdot q_{-k+1}\,. \tag{14}$$

Using (13), it is straightforward, though still some work, to show that indeed, for (12) to hold, every one of the vectors w_j corresponding to a $\left|\lambda_j\right| > \xi_k$ and to a $\phi_k P_{\cdot j} \neq 0$ must satisfy (11).

Of course a problem must have special structure in order for it to turn out that there is a k, j pair such that $\phi_k P_{\cdot j} = 0$. This is the justification for the (potentially misleading) common practice of assuming that if any linear combination of y's is constrained to grow slower than ξ^t, then all roots exceeding ξ_i in absolute value must be suppressed in the solution. If (11) does hold for all t then we can see from (10) that this entails

$$P^{j\cdot}\cdot(\Psi z + \Pi\eta) = 0\,. \tag{15}$$

Collecting all the rows of P^{-1} corresponding to j's for which (15) holds into a single matrix $P^{U\cdot}$ (where the U stands for "unstable"), we can write

$$P^{U\cdot}(\Psi z + \Pi\eta) = 0\,. \tag{16}$$

Existence problems arise if the endogenous shocks η cannot adjust to offset the exogenous shocks z in (16). We might expect this to happen if $P^{U\cdot}$ has more rows than has columns. This accounts for the usual notion that there are existence problems if the number of unstable roots exceeds the number of "jump variables". However, the precise condition is that columns of $P^{U\cdot}\Pi$ span the space spanned by the columns of $P^{U\cdot}\Psi$, i.e.

$$\operatorname{span}\left(P^{U\cdot}\Psi\right) \subset \operatorname{span}\left(P^{U\cdot}\Pi\right) . \tag{17}$$

In order for the solution to be unique, it must be that (16) pins down not only the value of $P^{U\cdot}\Pi\eta$, but also all the other error terms in the system that are influenced by η. That is, from knowledge of $P^{U\cdot}\Pi\eta$ we must be able to determine $P^{S\cdot}\Pi\eta$, where $P^{S\cdot}$ is made up of all the rows of P^{-1} not included in $P^{U\cdot}$. Formally, the solution is unique if and only if

$$\operatorname{span}\left(\Pi'\left(P^{S\cdot}\right)'\right) \subset \operatorname{span}\left(\Pi'\left(P^{U\cdot}\right)'\right) . \tag{18}$$

In this case we will have

$$P^{S\cdot}\Pi\eta = \Phi P^{U\cdot}\Pi\eta \tag{19}$$

for some matrix Φ.

Usually we aim at writing the system in a form that can be simulated from arbitrary initial conditions, delivering a solution path that does not violate the stability conditions. We can construct such a system by assembling the equations of the form delivered by the stability conditions (11), together with the lines of (10) that determine w_S, the components of w not determined by the stability conditions, and use (16) to eliminate dependence on η. Specifically, we can use the system

$$\begin{bmatrix} w_S(t) \\ w_U(t) \end{bmatrix} = \begin{bmatrix} \Lambda_S \\ 0 \end{bmatrix} w_S(t-1) + \begin{bmatrix} P^{S\cdot}C \\ (I-\Lambda_U)^{-1} P^{U\cdot}C \end{bmatrix} + \begin{bmatrix} I & -\Phi \\ 0 & 0 \end{bmatrix} P^{-1}\Psi z . \tag{20}$$

To arrive at an equation in y, we use $y = Pw$ to transform (20) into

$$\begin{aligned} y(t) = P_{\cdot S}\Lambda_S P^{S\cdot} y(t-1) + \left(P_{\cdot S}P^{S\cdot} + P_{\cdot U}\left(I-\Lambda_U\right)^{-1} P^{U\cdot}\right) C \\ + \left(P_{\cdot S}P^{S\cdot} - P_{\cdot S}\Phi P^{U\cdot}\right)\Psi z . \end{aligned} \tag{21}$$

Labeling the three matrix coefficients in (21), we can give it the form

$$y(t) = \Theta_1 y(t-1) + \Theta_c C + \Theta_z z(t) , \tag{22}$$

which can in turn be used to characterize the impulse responses of y, according to

$$y(t+s) - E_t y(t+s) = \sum_{v=0}^{s-1} \Theta_1^v \Theta_z z(t+s-v) . \tag{23}$$

Of course to completely characterize the mapping from initial conditions and z realizations to y, we need in addition to (23) a formula for the predictable part of y, i.e.

$$E_t y(t+s) = \Theta_1^s y(t) + (I - \Theta_1^{s+1}) \cdot (I - \Theta_1)^{-1} \Theta_C C . \tag{24}$$

However all the information needed to compute both (23) and (24) is contained in a report of the coefficient matrices for (21). Note that (21), while it insures that the second row of (20),

$$w_U(t) = P^{U\cdot} y(t) = (I - \Lambda_U)^{-1} P^{U\cdot} C , \tag{25}$$

holds for all t after the initial date $t = 0$, it does not in itself impose (25) at $t = 0$, which in fact is required by the solution.

3.2. **Continuous time.** In this type of canonical form, the analysis for continuous time is nearly identical to that for discrete time. The equation we start with is

$$\dot{y} = \Gamma_1 y + C + \Psi z + \Pi \eta , \tag{26}$$

where z and η are both assumed to be derivatives of martingale processes, i.e. white noise. Just as in discrete time, we form a Jordan decomposition of Γ_1 of the form (7). Again we use it to change variables to $w = P^{-1} y$ and split w into components w_S and w_U that need not be suppressed, and need to be suppressed, respectively, to satisfy the stability conditions. The stability conditions in continuous time are naturally written, analogously to (12),

$$E_s [\phi_k y(t)] e^{-\xi_k t} = \phi_k P E_s [w(t)] e^{-\xi_k t} = \phi_k P e^{\Lambda \cdot (t-s)} \left[w(s) - \Lambda^{-1} P^{-1} C \right] e^{-\xi_k t} \to 0 . \tag{27}$$

The restricted components of w are then those that correspond both to a non-zero $\phi_k P^{\cdot j}$ and to a λ_j with real part exceeding the corresponding ξ_k. Once we have partitioned w, P, and P^{-1} into S and U components according to this criterion, the analysis proceeds as in the discrete case, with the conditions for existence and uniqueness applying in unchanged form — (17) and (18) . The final form of the equation usable for simulation is

$$\dot{y} = P_{\cdot S} \Lambda_S P^{S\cdot} y + P_{\cdot S} P^{S\cdot} C + P_{\cdot S} \left(P^{S\cdot} - \Phi P^{U\cdot} \right) \Psi z . \tag{28}$$

This equation is usable to compute impulse responses, according to

$$y(t+s) - E_t y(t+s) = \int_0^s e^{\Theta_1 v} \Theta_z z(t+s-v) \, ds \tag{29}$$

and the deterministic part of y, according to

$$E_t y(t+s) = e^{\Theta_1 s} y(t) - \left(I - e^{\Theta_1 s} \right) \Theta_1^{-1} \Theta_c C . \tag{30}$$

In this case, in contrast to the discrete time case, the preceding three equations contain no information about the stability conditions restricting the levels of y at all. We need to specify separately the condition

$$w_U(t) = P^{U\cdot} y(t) = -\Lambda_U^{-1} P^{U\cdot} C \tag{31}$$

4. DISCRETE TIME, SOLVING FORWARD

In this section we consider the generic canonical form (1), allowing for possibly singular Γ_0 and not requiring z to be serially uncorrelated. At first, though, we consider only the case where there is a single bound $\bar{\xi}$ on the maximal growth rate of any component of y. We find conditions that prevent such explosive growth as follows. First we compute a QZ decomposition

$$\begin{aligned} Q'\Lambda Z' &= \Gamma_0 \\ Q'\Omega Z' &= \Gamma_1 \,. \end{aligned} \tag{32}$$

In this decomposition, $Q'Q = Z'Z = I$, where Q and Z are both possibly complex and the $'$ symbol indicates transposition and complex conjugation. Also Ω and Λ are possibly complex and are upper triangular. The QZ decomposition always exists. Letting $w(t) = Z'y(t)$, we can multiply (1) by Q to obtain

$$\Lambda w(t) = \Omega w(t-1) + QC + Q\Pi\eta(t) + Q\Psi z(t)\,. \tag{33}$$

Though the QZ decomposition is not unique, the collection of values for the ratios of diagonal elements of Ω and Λ, $\{\omega_{ii}/\lambda_{ii}\}$ (called the set of generalized eigenvalues), is usually unique (if we include ∞ as a possible value). The generalized eigenvalues are indeterminate only when Γ_0 and Γ_1 have zero eigenvalues corresponding to the same eigenvector.[1] We can always arrange to have the largest of the generalized eigenvalues in absolute value appear at the lower right. In particular, let us suppose that we have partitioned (8) so that $|\omega_{ii}/\lambda_{ii}| \geqslant \bar{\xi}$ for all i>k and $|\omega_{ii}/\lambda_{ii}| < \bar{\xi}$ for all $i \le k$. Then (8) can be expanded as

$$\begin{aligned} &\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \cdot \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \cdot \begin{bmatrix} w_1(t-1) \\ w_2(t-1) \end{bmatrix} \\ &+ \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} (C + \Psi z(t) + \Pi\eta(t)) \end{aligned} \tag{34}$$

Note that some diagonal elements of Λ_{22}, but not of Λ_{11}, may be zero. Also note that zeros at the same position i on the diagonals of both Λ and Ω cannot occur unless the equation system is incomplete, meaning that some equation is exactly a linear combination of the others.

[1] This would imply that a linear combination of the equations contains no y's, i.e. that there is effectively one equation fewer for determining the y's than would appear from the order of the system.

Because of the way we have grouped the generalized eigenvalues, the lower block of equations in (33) is purely explosive. It has a solution that does not explode any faster than the disturbances z so long as we solve it "forward" to make w_2 a function of future z's. That is, if we label the last additive term in (34) $x(t)$ and set $M = \Omega_{22}^{-1} \cdot \Lambda_{22}$,

$$\begin{aligned} Z'_{\cdot 2} y(t) = w_2(t) &= M w_2(t+1) - \Omega_{22}^{-1} x_2(t+1) \\ &= M^2 \cdot w_2(t+2) - M \cdot \Omega_{22}^{-1} \cdot x_2(t+2) - \Omega_{22}^{-1} \cdot x_2(t+1) \\ &= -\sum_{s=1}^{\infty} M^{s-1} \cdot \Omega_{22}^{-1} \cdot x_2(t+s)\,. \end{aligned} \tag{35}$$

The last equality in (35) follows on the assumption that $M^t w_2(t) \to 0$ as $t \to \infty$. Note that in the special case of $\lambda_{ii} = 0$ there are equations in (34) containing no current values of w. While these cases do not imply explosive growth, the corresponding components of (35) are still valid. For example, if the lower right element of Λ is zero, the last equation of (34) has the form

$$0 \cdot w_n(t) = \omega_{nn} \cdot w_n(t-1) + x_n(t). \tag{36}$$

Solving for $w_n(t-1)$ produces the corresponding component of (35). Since $\lambda_{ii} = 0$ corresponds to a singularity in Γ_0, the method we are describing handles such singularities transparently.

Note that (35) asserts the equality of something on the left that is known at time t to something on the right that is a combination of variables dated $t+1$ and later. Since taking expectations as of date t leaves the left-hand side unchanged, we can write

$$\begin{aligned} Z'_{\cdot 2} y(t) = w_2(t) &= -E_t \left[\sum_{s=1}^{\infty} M^{s-1} \cdot \Omega_{22}^{-1} \cdot x_2(t+s) \right] \\ &= -\sum_{s=1}^{\infty} M^{s-1} \cdot \Omega_{22}^{-1} \cdot x_2(t+s)\,. \end{aligned} \tag{37}$$

If the original system (1) consisted of first order conditions from an optimization problem, (37) will be what is usually called the decision rule for the problem. When the original system described a dynamic market equilibrium, (37) will be composed of decision rules of the various types of agents in the economy, together with pricing functions that map the economy's state into a vector of prices.

The last equality in (37) imposes conditions on x_2 that may or may not be consistent with the economic interpretation of the model. Recall that x_2 is made up of constants, terms in z, and terms in η. But z is an exogenously given stochastic processes whose properties cannot be taken to be related to Γ_0 and Γ_1. Thus if it should turn out that x_2 contains no η component, the assertion in (37) will be impossible — it requires that exogenously evolving events that occur in the future be known precisely now. If x_2 does contain an η

component, then (37) asserts that this endogenously determined component of randomness must fluctuate as a function of future z's so as exactly to prevent any deviation of the right-hand side of (37) from its expected value.

Replacing x's in (37) with their definitions, that equation becomes

$$\begin{aligned} Z'_{\cdot 2}y(t) &= (\Lambda_{22}-\Omega_{22})^{-1}Q_{2\cdot}C - E_t\left[\sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi z(t+s)\right] \\ &= (\Lambda_{22}-\Omega_{22})^{-1}Q_{2\cdot}C - \sum_{s=1}^{\infty} M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\left(\Psi z(t+s)+\Pi\eta(t+s)\right). \quad (38) \end{aligned}$$

The latter equality is satisfied if and only if

$$Q_{2\cdot}\Pi\eta(t+1) = \sum_{s=1}^{\infty}\Omega_{22}M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi\cdot\left(E_{t+1}z(t+s)-E_t z(t+s)\right). \quad (39)$$

A leading special case is that of serially uncorrelated z's, i.e. $E_t z(t+s)=0$, all $s>1$. In this case the right-hand-side of (39) is just $Q_{2\cdot}\Psi z(t+1)$, so a necessary condition for the existence of a solution[2] satisfying (39) is that the column space of $Q_{2\cdot}\Psi$ be contained in that of $Q_{2\cdot}\Pi$, i.e.

$$\operatorname{span}(Q_{2\cdot}\Psi) \subset \operatorname{span}(Q_{2\cdot}\Pi). \quad (40)$$

This condition is necessary and sufficient when $E_t z(t+1)=0$, all t. A necessary and sufficient condition for a solution to exist regardless of the pattern of changes in the expected future time path of z's is[3]

$$\operatorname{span}\left(\left\{\Omega_{22}M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi\right\}_{s=1}^{n-k}\right) \subset \operatorname{span}(Q_{2\cdot}\Pi). \quad (41)$$

In most economic models this latter necessary and sufficient condition is the more meaningful one, even if it is always true that $E_t z(t+1)=0$, because ordinarily our theory places no reliable restrictions on the serial dependence of the z process, even if we have made some assumption on it for the purpose at hand.

Assuming a solution exists, we can combine (39), or its equivalent in terms of w (35), with some linear combination of equations in (34) to obtain a new complete system in w that is stable. However, the resulting system will not be directly useful for generating

[2]Note that here it is important to our analysis that there are no hidden restrictions on variation in z that cannot be deduced from the structure of the equation system. In an equation system in which there are two exogenous variables with $z_1(t)=2z_2(t-2)$, for example, our analysis requires that this restriction connecting the two exogenous variables be included as an equation in the system and the number of exogenous variables be reduced to one.

[3]It may appear that s should range from 1 to infinity, rather than 1 to $n-k$, in this expression. But $\Omega_{22}^{-1}Q_{2\cdot}\Psi$ is in some invariant subspace of M, if only the trivial full $n-k$-dimensional space Euclidean space. The invariant space containing it, say of dimension j, will be spanned by j elements of the $M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi$ sequence.

simulations or distributions of y from specified processes for z unless we can free it from references to the endogenous error term η. From (39), we have an expression that will determine $Q_{2\cdot}\Pi\eta(t)$ from information available at t and a known stochastic process for z. However the system also involves a different linear transformation of η, $Q_{1\cdot}\Pi\eta(t)$. It is possible that knowing $Q_{2\cdot}\Pi\eta(t)$ is not enough to tell us the value of $Q_{1\cdot}\Pi\eta(t)$, in which case the solution to the model is not unique. In order that the solution be unique it is necessary and sufficient that the row space of $Q_{1\cdot}\Pi$ be contained in that of $Q_{2\cdot}\Pi$. In that case we can write

$$Q_{1\cdot}\Pi = \Phi Q_{2\cdot}\Pi \tag{42}$$

for some matrix Φ. Premultiplying (34) by $[I \quad -\Phi]$ gives us a new set of equations, free of references to η, that can be combined with (35) to give us

$$\begin{aligned}
&\begin{bmatrix}\Lambda_{11} & \Lambda_{12}-\Phi\Lambda_{22}\\ 0 & I\end{bmatrix}\cdot\begin{bmatrix}w_1(t)\\ w_2(t)\end{bmatrix}\\
&\quad=\begin{bmatrix}\Omega_{11} & \Omega_{12}-\Phi\Omega_{22}\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}w_1(t-1)\\ w_2(t-1)\end{bmatrix}+\begin{bmatrix}Q_{1\cdot}-\Phi Q_{2\cdot}\\ (\Omega_{22}-\Lambda_{22})^{-1}Q_{2\cdot}\end{bmatrix}C\\
&\qquad+\begin{bmatrix}Q_{1\cdot}-\Phi Q_{2\cdot}\\ 0\end{bmatrix}\Psi z(t)-E_t\begin{bmatrix}0\\ \sum_{s=1}^{\infty}M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi z(t+s)\end{bmatrix}.
\end{aligned}\tag{43}$$

This can be translated into a system in y of the form

$$y(t) = \Theta_1 y(t-1) + \Theta_c + \Theta_0 z(t) + \Theta_y \sum_{s=1}^{\infty} \Theta_f^{s-1}\Theta_z E_t z(t+s)\,. \tag{44}$$

The details of the translation are

$$\begin{gathered}
H = Z\begin{bmatrix}\Lambda_{11}^{-1} & -\Lambda_{11}^{-1}(\Lambda_{12}-\Phi\Lambda_{22})\\ 0 & I\end{bmatrix};\quad \Theta_1 = Z_{\cdot 1}\Lambda_{11}^{-1}\left[\Omega_{11} \quad (\Omega_{12}-\Phi\Omega_{22})\right]Z;\\
\Theta_c = H\cdot\begin{bmatrix}Q_{1\cdot}-\Phi Q_{2\cdot}\\ (\Omega_{22}-\Lambda_{22})^{-1}Q_{2\cdot}\end{bmatrix}C;\quad \Theta_0 = H\cdot\begin{bmatrix}Q_{1\cdot}-\Phi Q_{2\cdot}\\ 0\end{bmatrix}\cdot\Psi;\\
\Theta_y = -H_{\cdot 2};\quad \Theta_f = M;\quad \Theta_z = \Omega_{22}^{-1}Q_{2\cdot}\Psi\,.
\end{gathered}\tag{45}$$

The system defined by (44) and (45) can always be computed and is always a complete equation system for y satisfying the condition that its solution grow slower than $\bar{\xi}^t$, even if there is no solution for η in (39) or the solution for $Q_{1\cdot}\Pi$ in (42) is not unique. If there is no solution to (39), then (44)-(45) implicitly restricts the way z enters the system, achieving stability by contradicting the original specification. If the solution to (42) is not unique, then the absence of η from (44)-(45) contradicts the original specification. If the solution is not unique, but Φ is computed to satisfy (42), the (44)-(45) system generates one of the multiple solutions to (1) that grows slower that $\bar{\xi}^t$. If one is interested in generating the full set of non-unique solutions, one has to add back in, as additional “disturbances", the components of $Q_{1\cdot}\Pi\eta$ left undetermined by (42).

To summarize, we have the following necessary and sufficient conditions for existence and uniqueness of solutions satisyfing the bounded rate of growth condition:

(A) A necessary and sufficient condition that (1) has a solution meeting the growth condition for arbitrary assumptions on the expectations of future z's is that the column space spanned by

$$\left\{\Omega_{22}M^{s-1}\Omega_{22}^{-1}Q_{2\cdot}\Psi\right\}_{s=1}^{n-k}$$

be contained in that of $Q_{2\cdot}\Pi$.

(B) A necessary and sufficient condition that any solution to (1) be unique is that the row space of $Q_{1\cdot}\Pi$ be contained in that of $Q_{2\cdot}\Pi$.

Condition A takes a simpler form if the system has fully specified the dynamics of exogenous variables:

A$'$. A necessary and sufficient condition that (1) has a solution meeting the growth condition for arbitrary assumptions on the covariance matrix of serially uncorrelated z's is (40), i.e. that the column space spanned by $Q_{2\cdot}\Psi$ be contained in that of $Q_{2\cdot}\Pi$.

When condition A is met, a solution is defined by (44)-(45). In the special case of $E_t z(t+1) = 0$, the last term of (44) (involving Θ_y, Θ_f and Θ_z) drops out.

5. Computational details

If A has full column rank, we can check to see whether the column space of a matrix A includes that of a matrix B by “regressing" B on A to see if the residuals are zero, i.e. by checking

$$\left(I - A(A'A)^{-1}A'\right)B = 0\,. \tag{46}$$

If A has full row rank, then its column space automatically includes any other space of the same dimension. If A has neither full row nor full column rank, other methods are required. The singular value decomposition (svd) of a matrix A is a representation

$$A = UDV' \tag{47}$$

in which U and V satisfy $U'U = I = V'V$ (but are not in general square) and D is square and diagonal.[4] If B's svd is TCW, A's column space includes B's if and only if

$$\left(I - UU'\right)T = 0\,. \tag{48}$$

[4]This is not actually the standard version of the svd. Matlab returns with U and V square, D of the same order as A. From such an svd, the form discussed in the text can be obtained by replacing D with a square non-singular matrix that has the non- zero diagonal elements of the original D on the diagonal and by forming U and V from the columns of the original U and V corresponding to non-zeros on the diagonal of the original D.

If (48) holds, then

$$B = A \cdot VD^{-1}U'B\,. \tag{49}$$

Equation (48) gives us a computationally stable way to check the column and row space spanning conditions summarized in A and B at the end of the previous section, and (49) is a computationally stable way to compute the matrix transforming A to B, and thus to compute the Φ matrix in (42).[5]

Though the QZ decomposition is available in Matlab, it emerges with the generalized eigenvalues not sorted along the diagonals of Λ and Ω. Since our application of it depends on getting the unstable roots into the lower right corner, an auxiliary routine is needed to sort the roots around a $\bar{\xi}$ level.[6]

If Γ_0 in (1) is invertible we can multiply through by its inverse to obtain a system which, like that in section 3, has $\Gamma_0 = I$. In such a system the QZ decomposition delivers $Q = Z'$, and the decomposition $\Gamma_1 = Q'\Omega Q$ is what is known as the (complex) Schur decomposition of the matrix Γ_1. Since the Schur decomposition is somewhat more widely available than the QZ, this may be useful to know. Also in such a system we may be able to use an ordinary eigenvalue decomposition of Γ_1 in place of either the Schur or the QZ. Most such routines return a matrix V whose columns are eigenvectors of Γ_1, together with a vector λ of eigenvalues. If V turns out to be non-singular, as it will if Γ_1 has no repeated roots, then $\Gamma_1 = V\,\mathrm{diag}(\lambda)V^{-1}$, and this decomposition can be used to check existence and uniqueness and to find a system of the form (44) that generates the stable solutions.

With V, V^{-1} and λ partitioned to put the excessively explosive roots in the lower right, we can write the conditions for existence and uniqueness just as in A and B of the previous section but setting the matrices in those conditions to the following:

$$\Omega_{22} = \mathrm{diag}(\lambda_{22})\,; \quad M = \mathrm{diag}(\lambda_{22}^{-1})\,; \quad Q_{2\cdot} = V^{2\cdot}\,; \quad Q_{1\cdot} = V^{1\cdot}\,. \tag{50}$$

Here $V^{i\cdot}$ refers to the i'th block of rows of V^{-1}. The calculations in (45) can be carried out based on (50) also, with $\Lambda = I$. The advantage of this approach to the problem is that, even though (50) is written with V, V^{-1} and λ ordered in a particular way, if the roots to be re-ordered are distinct, they can be re-ordered simply by permuting the elements of λ, the columns of V, and the rows of V^{-1}. There is no need here, as there is with QZ, to recompute elements of the matrices when the decomposition is re-ordered.

An intermediate form of simplification is available when Γ_0 is singular, but the generalized eigenvalues are all distinct. In that case it is possible to diagonalize Λ and Ω in (32) to

[5] Of course since Φ is applied to rows rather than columns, corresponding adjustments have to be made in the formula.

[6] The routine `qzdiv` does this and is available, with other Matlab routines that implement the methods of this paper, at `http://www.princeton.edu/~sims/#gensys`.

arrive at

$$\begin{aligned} P\Lambda_0 R &= \Gamma_0 \\ P\Lambda_1 R &= \Gamma_1 \end{aligned} \tag{51}$$

in which the Λ_i are diagonal and P and R are both nonsingular. Here the conditions, which we omit in detail, are almost exactly as in the QZ case, but as with (50), any re-ordering of the decomposition that may be required can be done by permutations rather than requiring new calculations.

6. More general growth rate conditions

Properly formulated dynamic models with multiple state variables generally do not put a uniform growth rate restriction on all the equations of the system. Transversality conditions, for example, usually restrict the ratio of a wealth variable to marginal utility not to explode. In a model with several assets, there is no constraint that individual assets not exponentially grow or shrink, so long as they do so in such a way as to leave total wealth satisfying the transversality condition. Ignoring this point can lead to missing sources of indeterminacy.

No program that simply calculates roots and counts how many fall in various partitions will be able to distinguish such cases. To handle conditions like these formally, we proceed as follows. We suppose that boundary conditions at infinity are given in the form

$$\xi_i^{-t} H_i y(t) \underset{t\to\infty}{\longrightarrow} 0\,, \tag{52}$$

$i = 1, \ldots, p$. Here $H_i y$ is the set of linear combinations of y that are constrained to grow slower than ξ_i^t. Suppose we have constructed the QZ decomposition (32) for our system and the j'th generalized eigenvalue ω_{jj}/λ_{jj} exceeds ξ_i in absolute value. To see whether this root needs to be put in the forward part of the solution or instead belongs in the backward part – i.e. to see whether the boundary condition generates a restriction – we must re-order the QZ decomposition so that the j'th root appears at the upper left. Then we can observe that

$$H_i y = H_i Z Z' y = H_i Z w\,, \tag{53}$$

where $w = Z'y$ as in (8). Assuming that no component of w other than the first produces explosive growth in $H_i y$ faster than ξ_i^t, the first component produces such growth if and only if the first column of $H_i Z$ is non-zero. If this first column is non-zero, the j'th root generates a restriction, otherwise it does not. A test of this sort, moving the root in question to the upper left of the QZ decomposition, needs to be done for each root that exceeds in absolute value any of the ξ_i's. Roots can be tested this way in groups, with a block of roots moved to the upper left together, and if it turns out that the block generates no restrictions, each one of the roots generates no restrictions. However if the block of roots generates restrictions, each root must then be tested separately to see whether it generates restrictions by itself. The only exception is complex pairs of roots, which in a real system should generate restrictions or not, jointly.

When we have this type of boundary condition, there may be a particular appeal to trying to achieve the decomposition (51), as with that decomposition the required re-orderings become trivial. Note that all that is essential to avoiding the work of re-ordering is that the generalized eigenvalues that are candidates for violating boundary conditions all be distinct and distinct from the eigenvalues that are not candidates for violating boundary conditions. In that case we can achieve a block diagonal version of (51), with all the roots not candidates for violating boundary conditions grouped in one block and the other roots entering separately on the diagonal.

The Matlab programs `gensys` and `gensysct` that accompany this paper do not automatically check the more general boundary conditions discussed in this section of the paper. Each has a component, however, called respectively `rawsys` and `rawsysct`, that computes the forward and backward parts of the solution from a QZ decomposition that has been sorted with all roots that generate restrictions grouped at the lower right. There is also a routine available, called qzswitch, that allows the user to re-order the QZ in any desired way. With these tools, it is possible to check existence and uniqueness and find the form of the solution when boundary conditions take the form (52).

7. Extension to a general continuous time system

Consider a system in continuous time[7] of the form

$$\Gamma_0 \dot{y} = \Gamma_1 y + C + \Psi z + \Pi \eta \,, \tag{54}$$

with an endogenous white noise error η and z an exogenous process that may include a white noise component with arbitrary covariance matrix. By a "white noise" here we mean the time-derivative of a martingale. The martingales corresponding to z and could have jump discontinuities in their paths without raising problems for this paper's analysis. A fully general analysis of continuous-time models is messier than for discrete-time models, because when z is an arbitrary exogenous process, the solution may in general contain both a “forward component" like that in the discrete time case and a “differential component" that relates y to the non-white-noise components of first and higher- order derivatives of z. We therefore first work through the case of white-noise z, which is only a slight variation on the analysis for the discrete-time model with serially uncorrelated z.

[7]In this section we assume the reader is familiar with the definition of a continuous time white noise and understands how integrals of them over time can be used to represent continuous time stochastic processes.

An example of such a system is the continuous time analogue of (2),

$$\begin{aligned} w(t) &= .3E_t \int_{s=0}^{\infty} e^{-.3s} W(t+s)\,ds - \alpha\cdot(u(t)-u_n) + \nu(t) \\ W(t) &= .3 \int_{s=0}^{\infty} e^{-.3s} w(t-s)\,ds \\ \dot{u}(t) &= -\theta\cdot u(t) + \gamma\cdot W(t) + \mu + \varepsilon(t)\,, \end{aligned} \tag{55}$$

which can be rewritten as

$$\begin{aligned} \dot{w} &= .3w - .3W - \alpha\dot{u} + .3\alpha\cdot(u-u_n) + z_1 - .3\nu + \eta_1 \\ \dot{\nu} &= z_1 \\ \dot{W} &= -.3W + .3w \\ \dot{u} &= -\theta u + \gamma W + \mu + z_2\,. \end{aligned} \tag{56}$$

Note that to fit the framework with all exogenous disturbances white noise, we have made ν a martingale. We could make other assumptions instead on the process generating ν, but to stay in this simple framework we need an explicit form for the process. In the notation of (54), (56) has

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \Gamma_1 = \begin{bmatrix} .3 & -.3 & -.3 & .3\alpha \\ 0 & 0 & 0 & 0 \\ .3 & 0 & -.3 & 0 \\ 0 & 0 & \gamma & -\theta \end{bmatrix},$$

$$C = \begin{bmatrix} -.3\alpha u_n \\ 0 \\ 0 \\ \mu \end{bmatrix},\ \Psi = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ \Pi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can proceed as before to do a QZ decomposition of Γ_0 and Γ_1 using the notation of (32), arriving at the analogue of (8)

$$\Lambda\dot{w} = \Omega w + QC + Q\Pi\eta + Q\Psi z\,. \tag{57}$$

Again we want to partition the system to arrive at an analogue to (34), but now instead of putting the roots largest in absolute value in the lower right corner of Ω, we want to put there the roots with algebraically largest (most positive) real parts.

Cases where Λ has zeros on the diagonal present somewhat different problems here than in the discrete case. In the discrete case, we can think of the ratio of a non-zero Ω_{ii} to a zero Λ_{ii} as infinite, which is surely very large in absolute value, and then treat these pairs as corresponding to explosive roots. Here, on the other hand, we are not concerned with the absolute values of roots, only with their real parts. The ratio of a non- zero value to zero cannot be treated as having a well-defined sign on its real part. Nonetheless, it turns out that we want to treat zeros on the diagonal of Λ as producing “unstable" roots.

If λ_{ii} is close to zero and ω_{ii}/λ_{ii} has positive real part, there is no doubt that we classify the corresponding generalized eigenvalue as unstable. If ω_{ii}/λ_{ii} has negative, extremely large real part, or a negative real part of any size combined with an extremely large imaginary part, we may still be justified in treating it as an unstable root. This is true as a matter of numerical analysis because extremely small λ_{ii} values may differ from zero only by accumulated rounding error. But also as a matter of economic interpretation, extremely large generalized eigenvalues correspond to components of y that, though they are technically defined as random variables at each date, behave so erratically that they approach white-noise like behavior.

We now proceed with our plan to partition (57) to obtain an analogue to (34). In the lower right we place all cases of λ_{ii} extremely close to zero, then just above that all cases of ω_{ii}/λ_{ii} positive and exceeding some boundary level $\bar{\xi} \geqslant 0$. The resulting system is

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & \Lambda_{22} & \Lambda_{23} \\ 0 & 0 & \Lambda_{33} \end{bmatrix} \cdot \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ 0 & \Omega_{22} & \Omega_{23} \\ 0 & 0 & \Omega_{33} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \\ Q_{3\cdot} \end{bmatrix} (C + \Psi z + \Pi\eta) . \quad (58)$$

The last block of equations in (58) can be written as

$$w_3 = \Omega_{33}^{-1} \left(\Lambda_{33} \dot{w}_3 - Q_{3\cdot} (C + \Psi z + \Pi\eta) \right) . \quad (59)$$

Because we require that w_3 be a random variable observable at t, $E_t w_3(t) = w_3(t)$. For white-noise z and η, $E_t z(t) = E_t \eta(t) = 0$. Since Λ_{33} is upper triangular with zeros on the diagonal, its bottom row is zero. Thus we can see from (59) that the bottom element in the w vector, w_{3n}, satisfies $w_{3n} = -\omega_{nn}^{-1} Q_{3n\cdot} C$, where ω_{nn} is the lower right diagonal element of Ω. But now we can proceed recursively up to the second-to-last row of (59), etc. to conclude that in fact

$$\begin{aligned} w_3 &= -\Omega_{33}^{-1} Q_{3\cdot} C \\ Q_{3\cdot} (\Psi z + \Pi\eta) &= 0 . \end{aligned} \quad (60)$$

Proceeding to the second block of equations in (58), we see it is purely explosive at a growth rate exceeding $\bar{\xi}$, so that as in the case of the explosive component in the discrete time model, we must insist that it follows its unique non-explosive solution, which is simply a constant.

Thus the full set of stability conditions is

$$\begin{bmatrix} w_2 \\ w_3 \end{bmatrix} = - \begin{bmatrix} \Omega_{22} & \Omega_{12} \\ 0 & \Omega_{33} \end{bmatrix}^{-1} \begin{bmatrix} Q_{2\cdot} C \\ Q_{3\cdot} C \end{bmatrix} \quad (61)$$

$$\begin{bmatrix} Q_{2\cdot} \\ Q_{3\cdot} \end{bmatrix} (\Psi z + \Pi\eta) = 0 . \tag{62}$$

Just as in the discrete case, we can now translate the equations and stability conditions in terms of w back in to a stable equation in y, here taking the form

$$\dot{y} = \Theta_1 y + \Theta_c + \Theta_0 z . \tag{63}$$

As in the discrete case also, (62) will allow us, when conditions for existence are met, to write

$$Q_{1\cdot}\Pi\eta = \Phi \cdot \begin{bmatrix} Q_{2\cdot}\Pi \\ Q_{3\cdot}\Pi \end{bmatrix} \tag{64}$$

for some Φ. Letting the subscript u (for "unstable") refer to the range of indexes in blocks 2 and 3 of (58), and then letting the u subscript play the role of the 2 subscript in the formulas of (45), we get the correct answers for the coefficients of (63). The conditions for existence and uniqueness are exactly the same as in the discrete case without serial correlation in z, i.e. A$'$ and B.

8. CONTINUOUS TIME, UNRESTRICTED z

In discrete time, a first-order polynomial of the form $\Lambda + \Omega L$, in which Λ and Ω are upper triangular and for each i either $|\omega_{ii}/\lambda_{ii}| > 1$ or $\lambda_{ii} = 0$, $\omega_{ii} \neq 0$, always has a convergent "forward" inverse, in the form of the sum that appears in the right-hand side of (44). In continuous time, the operation analogous to inverting such an "unstable" $\Lambda + \Omega L$ is inverting a polynomial in the differential operator D of the form $\Lambda D + \Omega$ in which for every i, either the real part of ω_{ii}/λ_{ii} is negative or $\lambda_{ii} = 0$. If there are 0's on the diagonal of Λ in this case, the resulting inverse is not simply a "forward" operator in continuous time, with integrals replacing sums, but is a convolution of such operators with finite-order polynomials in the differential operator D. To be specific, if Λ is upper triangular and has zeros on its main diagonal, then

$$(I - \Lambda D)^{-1} = \sum_{s=0}^{n-1} \Lambda^s D^s , \tag{65}$$

where n is the order of the Λ matrix. This can be checked easily, and follows from the fact that for a Λ of this form it is guaranteed that $\Lambda^n = 0$. Unless the upper triangle of Λ is dense, it is likely that terms in (65) for larger s turn out to be zero.

If as before we group blocks 2 and 3 of (34) into a u block, still using 1 to label the other block, we can write that equation, using the differential operator D, as

$$\left(\begin{bmatrix} \Lambda_{11} & \Lambda_{1u} \\ 0 & \Lambda_{uu} \end{bmatrix} D - \begin{bmatrix} \Omega_{11}\Omega_{1u} \\ 0\Omega_{uu} \end{bmatrix} \right) \begin{bmatrix} w_1 \\ w_u \end{bmatrix} = \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} (C + \Psi z + \Pi\eta) . \tag{66}$$

Using the fact we observed in (65), we can see that the differential operator on the left-hand side of this equation, in the lower block, $\Lambda_u D - \Omega_{uu}$, has a stable inverse that is a convolution of a finite-order matrix polynomial in D with exponentially weighted averages of future values. While it is not worthwhile to write out the whole inverse explicitly, we can observe that the lower block can be solved recursively as

$$w_3 = -(I - \Omega_{33}^{-1}\Lambda_{33}D)^{-1}\Omega_{33}^{-1}Q_{3\cdot}(C + \Psi z(t)) \tag{67}$$

$$\begin{aligned} w_2(t) = -\int_0^\infty & e^{-\Lambda_{22}^{-1}\Omega_{22}s}\lambda_{22}^{-1}\cdot \\ & \big(Q_{2\cdot}(C + \Psi z(t+s) + \Pi\eta(t+s)) - \Lambda_{23}Dw_3(t+s) + \Omega_{23}w_3(t+s)\big)\,ds\,. \end{aligned} \tag{68}$$

The inverted polynomial in (67) becomes a finite order polynomial in D according to (65), and the D operators appearing on right of (68) are interpreted as applying to $E_t z(t+s)$, considered as a function of s and (at $s = 0$) taken as right-derivatives.

Existence and uniqueness questions are determined by exactly the same conditions as in conditions A and B for the discrete case, with the u subscript playing the role of the 2 subscript in those conditions. To get a complete solution, we combine the recursive solution for w_u in (68) and (67) with the stable equation in w_1, free of occurrences of η, that is obtained by multiplying (58) by $[I \quad -\Phi]$, where Φ has been computed to solve (42).

The continuous time version of the matlab software, `gensysct`, does not attempt to provide an explicit complete solution for this general case. It checks existence and uniqueness and provides the matrices needed to write out an explicit solution for the case of pure white noise z in the form (63), and it provides the QZ decomposition ordered as in (58).

Note that if one wants the two-sided (on future and past) projection of y on z, and if one is confident (perhaps because of having run `gensys`) that there are no existence or uniqueness problems, one can find the projection by Fourier methods directly. That is, one can form $\tilde{\Xi}(\omega) = i\omega\Gamma_0 - \Gamma_1$ and form the projection as the inverse Fourier transform of $\tilde{\Xi}^{-1}\Psi$, where the inverse in this expression is frequency-by-frequency matrix inversion. Of course this only produces stable results if the projection does not involve delta functions or derivative operators.

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