HIGH CONFIDENCE IMPLIES HIGH CREDIBILITY AND VICE VERSA

Suppose we have a confidence set at level α, so that

$$P[\beta \in C(X) \mid \beta] \geq 1 - \alpha \tag{1}$$

for all parameter values β in the parameter space S. From a Bayesian perspective, if the space of possible X values is U, then in the joint probability space $S \times U$, the prior pdf $\pi(\beta)$ on β lets us use the coverage probabilities $P[\beta \in C(X) \mid \beta]$ to form the probability of the set $Q \in S \times U$ defined as $Q = \{\beta, X \mid \beta \in C(X)\}$. Then it is clear that

$$P[Q] = \int P[Q \mid \beta]\pi(\beta)\, d\beta \geq 1 - \alpha. \tag{2}$$

This follows because the integral above is just a weighted average of coverage probabilities, all of which equal or exceed $1 - \alpha$. Let $q(X)$ be the marginal density for X, which is just

$$\int \pi(\beta)p(X \mid \beta)\, d\beta\,. \tag{3}$$

Then another way to write the unconditional $P[Q]$ is

$$P[Q] = \int P[Q \mid X]q(X)\, dX\,. \tag{4}$$

$P[Q \mid X]$ is just the Bayesian posterior probability on Q, and the expression above can be read as saying that the unconditional probability of Q is an average over X of credibility levels (posterior probabilities) for $C(X)$.

Let $R \subset U$ be the set of X values for which $P[Q \mid X] < 1 - \varepsilon$ and let $\gamma(\varepsilon)$ be $P[R]$. Then we can put an upper bound on $P[Q]$ via

$$1 - \alpha \leq P[Q] \leq (1 - \varepsilon)\gamma(\varepsilon) + 1 - \gamma(\varepsilon)\,. \tag{5}$$

Rearranging terms and subtracting 1 from both sides, we get

$$-\alpha \leq -\varepsilon\gamma(\varepsilon) \tag{6}$$

$$\therefore \alpha \geq \varepsilon\gamma(\varepsilon)\,. \tag{7}$$

The $\gamma(\varepsilon)$ function is obviously decreasing in ε, and $\gamma(1) = 0$, $\gamma(0) = 1$. Therefore if it is continuous, there is an ε such that $\gamma(\varepsilon) = \varepsilon$. At such a value of ε, (7) implies

$$\varepsilon \leq \sqrt{\alpha}\,.$$

Putting this result into words, if we have a confidence set with coverage probability everywhere at least $1 - \alpha$, then except on a set of X values with prior probability $\sqrt{\alpha}$, the posterior probability that β is in $C(X)$ will be at least $1 - \sqrt{\alpha}$.

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If there is a discontinuity, the argument has to get pickier, without changing the result or the basic intuition. In this case there might be no ε with $\varepsilon = \gamma(\varepsilon)$, but if not, because of the monotonicity, there does have to be a value of ε with

$$\gamma^+ = \lim_{\delta \downarrow 0} \gamma(\varepsilon - \delta) \geq \varepsilon \geq \gamma^- = \lim_{\delta \downarrow 0} \gamma(\varepsilon + \delta) .$$

But then $\gamma^+ \varepsilon \leq \alpha$, so $\varepsilon \leq \sqrt{\alpha}$. Since by assumption ε is strictly less than γ^+ and strictly greater than γ^-, we know that there is an $\varepsilon^+ > \varepsilon$ that still satisfies $\varepsilon < \sqrt{\alpha}$, for which $\gamma(\varepsilon^+) < \varepsilon < \sqrt{\alpha}$. This lets us conclude that for an $\varepsilon < \sqrt{\alpha}$, except on a set of X values with prior probability less than ε, the posterior probability that β is in $C(X)$ will be at least $1 - \varepsilon$.

Note that the confidence set you start with has α level many times smaller than the levels of ε and γ you can guarantee. E.g., with coverage probability of the confidence set at .01, you can only guarantee $\varepsilon = \gamma = .10$. To get $\varepsilon = \gamma = .01$, you would need to have a confidence set with $\alpha = .0001$.

As you should be able to see, this same argument can be run in reverse. If for every X we construct a set $C(X)$ that has posterior probability $P[C(X) \mid X]$ greater than or equal to $1 - \alpha$, then the probability γ of the set of β values for which the coverage probability $P[C(X) \mid \beta] < 1 - \varepsilon$ must satisfy $\alpha > \varepsilon\gamma$. This is the same formula as before, but now of course the interpretation of ε and γ are different.