EXERCISE ON CONDITIONAL, MARGINAL PROBABILITY, INDEPENDENCE, CHANGE OF VARIABLES.DUE FRIDAY 9/26 BY NOON.

(1) The exponential distribution, with pdf $\alpha e^{-\alpha\tau}$ on $\tau > 0$, is sometimes used as a distribution for "waiting times", e.g. for lengths of unemployment spells.

(a) Suppose we have a sample of N i.i.d. observations on unemployment durations τ_j, $j = 1, \dots, N$. If we treat the waiting times as exponential and our prior pdf for the parameter α is $\gamma e^{-\gamma\alpha}$ for some $\gamma > 0$, what is the conditional pdf for α given γ and the data?
It is proportional to

$$\alpha^N e^{(-\sum \tau_j - \gamma)\alpha} ,$$

which is a $\text{Gamma}(N + 1, \sum \tau_j + \gamma)$ pdf.

(b) Suppose we thought that actually different people have different α's, with the $\alpha's$ in the population distributed as $\beta e^{-\alpha\beta}$. We don't know the α_j's. What is the pdf of a randomly drawn τ_j conditional on β alone?
The joint pdf of τ_j, α_j is the conditional for $\tau_j \mid \alpha_j$ times the marginal on α_j, and we obtain the marginal for τ_j by integrating the joint w.r.t. α_j. I.e.,

$$p(\tau_j \mid \beta) = \int_0^\infty \beta e^{-\beta\alpha_j} \alpha_j e^{-\alpha_j \tau_j} \, d\alpha_j$$

$$= \int_0^\infty \beta \alpha_j e^{-(\beta+\tau_j)\alpha_j} \, d\alpha_j$$

$$= \frac{\beta}{(\beta + \tau_j)^2} .$$

The last equality follows because the joint pdf, as a function of α_j, is proportional to a $\text{Gamma}(2, \beta + \tau_j)$ pdf, but is not normalized to integrate to one. We can use this observation to see what the integral has to be by just looking up the constant in front of a $\text{Gamma}(2, \theta)$ pdf.

(c) Suppose we discover that our data is actually truncated. The people in the sample were only observed for 8 months. Spell lengths for those who found jobs during the 8 months were recorded, and all those spells still incomplete at the end of the 8 months were deleted from the sample. What is the pdf for a randomly drawn spell in this sample (assuming here a single α applies to all people.) [Hint: a) you want a conditional distribution given spell length less than 8 months and b) conditional pdf's integrate to one.]
The pdf of a single observation is

$$\frac{\alpha e^{-\alpha\tau_j}}{1 - e^{-8\alpha}} \quad \text{on } (0, 8) .$$

(2) Suppose x and y are jointly $N(0, I)$ and $\rho = \sqrt{x^2 + y^2}$ and $\theta = \arctan(x/y)$.
(a) What is the joint pdf of ρ and θ?
We need the Jacobian of the transform from x, y to ρ, θ. It is

$$\left|\frac{\partial(\rho,\theta)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{y}{y^2+x^2} & \frac{-x}{y^2+x^2} \end{vmatrix} = \rho^{-1} .$$

The full joint pdf then is

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{\rho^2}{2}\rho \, d\rho \, d\theta} \quad \text{on } \{\rho>0, \theta\in(-\pi,\pi)\} .$$

(b) What is the conditional pdf of $\rho \mid \theta = \pi/2$?
It is

$$\rho e^{-\frac{\rho^2}{2}} .$$

(c) What is the conditional pdf of $y \mid y \geq 0, x = 0$?

$$\sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}y^2} .$$

(d) Is the set $\{x, y \mid \theta = \pi/2\}$ the same as the set $\{x, y \mid x = 0, y \geq 0\}$? Is it true that within these two sets, $y = \rho$? Are the two conditional pdf's in (2b) and (2c) different? Does this make sense?
The sets are the same and within the set $y = \rho$. Nonetheless the two conditional pdf's differ. These conditional pdf's are not conditional on a *set*. They are conditional on σ-fields of sets. The geometric intuition is that dx corresponds to a very narrow, infinitely tall vertical rectangle, while $d\theta$ corresponds to a very thin pie-slice with its point at the origin. The fact that the pie-slice gets bigger as ρ increases is the reason that conditioning on $\theta = 0$ gives a conditional distribution for ρ that puts more weight on large ρ than does the distribution conditioning on $x = 0$.

(3) Suppose we have three random variables x, y and z^* that are jointly $N(0, I)$. Suppose we generate a fourth random variable z by the rule

$$z = \begin{cases} |z^*| & \text{if } xy > 0 \\ -|z^*| & \text{if } xy < 0 . \end{cases}$$

What is the marginal distribution of z? Does the marginal distribution of the pair x, z make them independent? Does the marginal distribution of the pair y, z make them independent? Are the three random variables x, y, z mutually independent?

The easy part is the last. They are certainly not independent, since once we know x and y we know the sign of z. In other words, the conditional distribution of $z \mid x, y$ does depend on x and y. $z \sim N(0,1)$. This is true, because it is with .5 probability a positive half-normal (pdf $2\phi(z)$ on $(0,\infty)$) and with .5 probability a negative half-normal. Such a mixture is obviously normal. What about the conditional distribution of $z \mid y$? It also is a 50% mixture of two half-normals, and thus is again $N(0,1)$, implying that the marginal distribution for y, z is a 2-dimensional $N(0,I)$. Similarly for the joint distribution of x, z. So we conclude that the variables are pairwise jointly normal and independent, but when considered as a three-dimensional vector are not jointly normal and not independent, though their covariance matrix is the identity.