EXERCISE ON PROBABILITY, DUE THURSDAY 9/19

(1) The state space S contains just four points $j = 1, \dots, 4$. Four securities with non-negative yields on the points in this space (and all linear combinations of them) are traded in a competitive, arbitrage-free market.

(a) Display a non-singular four by four matrix of yields on the four securities, with all entries positive, together with a vector of positive prices for them, that is impossible because it implies arbitrage opportunities. Show how you know that it presents an arbitrage opportunity.

Here's an answer:

$$y = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}, \qquad q = (2.5, 3.9, 1.7, 1.9),$$

where y is the matrix of security payoffs, each security in one column, states corresponding to rows. A security that pays 1 in state 4 and zero in the other states can be constructed as yw, where w is a column vector with elements

$$(0.275, -0.225, 0.025, 0.025).$$

Such a security, though, has price $q \cdot w$, which is -.1. Thus this set of yields and prices implies that it is possible to get a positive payoff, with no risk of loss, from a security that you will be paid to hold (instead of your having to pay to hold it). "Buying" arbitrarily large amounts of this thus gives you arbitrarily large resources now, when you're paid to take the security, and, if state 4 is realized, when the yields are realized.

How did I come up with this example? We know from class discussion that when linear combinations of securities can be bought and sold at constant prices, They must be priced as if there is a "market probability measure" that weights the states according to how likely they are (in a risk-adjusted sense). So I generated the y matrix, then formed a "bad" market measure that weighted the states as $(.1, .2, .8, -.1)$. This implied that the prices of the four securities were py, which is the vector shown as q above. That q came out all positive was lucky, as otherwise I would have had to keep experimenting, perhaps by replacing the negative-price security with a linear combination of it and a positive-price security. Of course the fact that my p vector had only one, slightly, negative price made the "luck" likely.

(b) Display a non-singular four by four matrix of yields on the four securities, with all entries non-negative, together with a vector of non-negative prices for them, that presents no arbitrage opportunities. Display the "market probabilities" and discount factor that are implied by the yield vectors and their prices.

Same y, but now $q = (2.4, 2.4, 2.4, 2.4)$. It looks pretty obvious that this will work, because of the symmetries in y — all the columns have the same sum. To calculate the market measure, we find $y^{-1}q = (.24, .24, .24, .24)$. Since the weights in this market measure sum to .86, we can represent it as a discount factor .86 times a market probability distribution that is uniform over the four states.

Though as usual you can collaborate on this exercise, if your matrices of yields in these answers exactly match those on another student's answers, one-half point (out of five) will be deducted from your grade. So randomize a bit.

(2) With S the state space, consisting of the four points $S = \{1, 2, 3, 4\}$, what is the smallest σ-field containing the two sets $\{2, 3, 4\}$ and $\{1, 2, 3\}$?

$\mathcal{F} = \left\{\emptyset, \{1\}, \{2,3,4\}, \{1,2,3\}, \{4\}, \{2,3\}, \{1,4\}, \{1,2,3,4\}\right\}$. The σ-field must contain the null set and the whole space, by the definition of a σ-field, and it must contain the two given subsets. Also the complements of the two given subsets, which are $\{1\}$ and $\{4\}$. Every subset of the space can be obtained by unions or intersections of these sets, except for subsets that contain only one of the elements of $\{2, 3\}$.

(3) With S the whole real line $S = \mathbb{R}$, does the smallest σ-field containing all intervals whose end points are finite rational numbers contain all intervals (a, b) where a and b are arbitrary real numbers, including $\pm\infty$? Explain your answer.

Yes. For any positive real number a, we can construct a sequence $\{a_j\}$ of rational numbers that converges to it — for example, we can truncate the real number's infinite decimal representation at successively higher numbers of decimal places. Any number with a finite decimal representation is rational, and obviously these "rounded" versions of a converge to a from below. To get a sequence of rationals that converge to a from above, use the inverses of the truncated decimal representations of $1/a$. And the same devices work to get sequences converging from above or below to negative numbers. By choosing $\{a_j\}$ to converge to a from above and $\{b_j\}$ to converge to b from below, we can represent any open interval (a, b) as $\bigcup_j (a_j, b_j)$, and thus any open interval is in the σ-field. Since there are obviously rational numbers converging to $\pm\infty$ (e.g. the integers), we can also represent any one-sided open interval (a, ∞) or $(-\infty, a)$ as a countable union of intervals with rational endpoints. So, since the intervals (a, b) with arbitrary real endpoints are contained within the σ-field with rational end points, the smallest σ-field containing either of them contains the smallest field containing the other.