ANSWERS TO HARD PARTS OF EXERCISE ON ASYMPTOTICS

(1) Suppose

$$\underset{T\times 1}{Y} \sim N(\underset{T\times 1}{\mathbf{1}}\mu, \Omega)$$

and that the elements of Y_t form a stationary process. This implies that the i'th row, j'th column of Ω is $\omega_{ij} = R(|i-j|)$, i.e. that Ω constant along every diagonal. Note that this is a special case of the usual GLS model, in which Ω has been given some structure and in which the explanatory variable X matrix is just a column of ones.

(c) Show that if we estimate $R(s)$ by the formula

$$\hat{R}_T(s) = \frac{1}{T}\sum_{v=s+1}^{T}(Y_t - \hat{\mu}_T)(Y_{t-s} - \hat{\mu}_T)\,,$$

where $\bar{Y}_T$ is the sample mean of $\{Y_t\}_1^T$, then $\hat{R}_T(s)$ is a consistent estimate of $R(s)$ for every s.

It is a fact that absolute summability of $R(s)$ is a sufficient condition for ergodicity for a Gaussian process. If you use that fact the argument is fairly simple. Proving the result directly from the assumptions would be a lot of work, as far as I can see, as it involves reproducing, for this special case, much of the logic of the theorem that characterizes conditions for ergodicity of a Gaussian process.

(e) Show that in this model, if we try to apply the formula that says the variance matrix of $\hat{\beta}_{OLS}$ is $(X'X)^{-1}X'\Omega X(X'X)^{-1}$ by substituting $\hat{\Omega}_T$ for Ω, we get nonsense. [Hint: An important computational fact about OLS estimates is that if $\hat{u}$ is the OLS residual vector and X the right-hand-side variable matrix, $X'\hat{u} = 0$.]

If this method is to work, it would have to be that

$$\liminf_{T\to\infty}\frac{1}{T}E[\mathbf{1}'\hat{\Omega}\mathbf{1}\mid X] \le \lim_{T\to\infty}\frac{1}{T}\mathbf{1}'\Omega\mathbf{1}\,.$$

(The random term, on the left, could converge in probability to the right-hand-term yet have a first moment that blows up. But since both terms are positive, the random term can't have a smaller expectation than the non-random term if the difference converges in probability to zero.) We will calculate the expectation for the special case where $\Omega = I$.

With $\Omega = I$, the non-random term is just $1/T$ times the sum of the elements of a $T\times T$ identity matrix, i.e. 1. The random matrix is $\mathbf{1}'Z_T'Z_T\mathbf{1}/T$. This is just

$1/T$ times the sum of squared elements of the vector $\mathbf{1}'Z_T'$. That vector is easily seen to be

$$1'Z_T' = [\hat{u}_1, \hat{u}_1 + \hat{u}_2, \ldots, -(\hat{u}_{T-1} + \hat{u}_T), -\hat{u}_T, 0, -\hat{u}_1, -(\hat{u}_1 + \hat{u}_2), \ldots, \hat{u}_{T-1} + \hat{u}_T, \hat{u}_T] \,,$$

where we are using the fact that the sum of the $\hat{u}_t$'s is zero. The t'th element of the first half of this vector is

$$\sum_1^t \hat{u}_t = \sum_{s=1}^t (1 - t/T) u_s - (t/T) \sum_{s=t+1}^T u_s \,.$$

The expected squared value of this term is

$$t(1 - t/T)^2 + (t/T)^2(T - t) \,.$$

Since the two halves of the $Z_T'\hat{u}$ vector are the same except for sign, the total length of the vector is

$$2 \left(\sum_{t=1}^{T-1} t \left(1 - \frac{t}{T} \right)^2 + \frac{t^2}{T^2}(T - t) \right) \,.$$

It is a fact that $\sum_1^T t = (T^2 + T)/2$ and $\sum_1^T t^2 = (2T^3 + 3T^2 + T)/6$. (I will not deliberately create an exam question where you have to have these formulas memorized. You can look them up in some calculus textbooks.) Using these facts, we can evaluate the displayed expression above as:

$$\frac{T^2 - 2T}{6} \,.$$

(It may seem that we also need a formula for $\sum t^3$, but the terms in t^3 cancel out.) If we multiply this expression by $2T^{-2}$, we have the expected value of $T^{-1}\mathbf{1}'\hat{\Omega}\mathbf{1}$, which should be one if it is to match $T^{-1}\mathbf{1}'\Omega\mathbf{1} \equiv 1$. But instead we see that the expression is always less than 1/3, converging to that value from below as $T \to \infty$.

This proves that plugging $\hat{\Omega}$ estimated this way into the standard formula does not give good results even asymptotically.