EXERCISE ON ASYMPTOTICS

(1) Suppose

$$\underset{T\times 1}{Y} \sim N(\underset{T\times 1}{\mathbf{1}}\mu, \Omega)$$

and that the elements of Y_t form a stationary process. This implies that the i'th row, j'th column of Ω is $\omega_{ij} = R(|i-j|)$, i.e. that Ω constant along every diagonal. Note that this is a special case of the usual GLS model, in which Ω has been given some structure and in which the explanatory variable X matrix is just a column of ones.

(a) Show that the OLS estimator of μ is the sample mean $\hat{\mu}_T$.
(b) Prove that if $\sum_s |R(s)|$ is finite, then $\hat{\mu}_T$ is a consistent estimator for μ.
(c) Show that if we estimate $R(s)$ by the formula

$$\hat{R}_T(s) = \frac{1}{T}\sum_{t=s+1}^{T} (Y_t - \hat{\mu}_T)(Y_{t-s} - \hat{\mu}_T)\,,$$

$\hat{R}_T(s)$ is a consistent estimate of $R(s)$ for every s.
(d) Show that if we form $\hat{\Omega}_T$ by setting $\hat{\omega}_{Tij} = \hat{R}_T(|i-j|)$, $\hat{\Omega}_T$ is positive semi-definite. [Hint: It can be written as $Z_T'Z_T$ for some choice of Z_T, with Z_T having more than T rows.]
(e) Show that in this model, if we try to apply the formula that says the variance matrix of $\hat{\beta}_{OLS}$ is $(X'X)^{-1}X'\Omega X(X'X)^{-1}$ by substituting $\hat{\Omega}_T$ for Ω, we get nonsense. [Hint: An important computational fact about OLS estimates is that if $\hat{u}$ is the OLS residual vector and X the right-hand-side variable matrix, $X'\hat{u} = 0$.]
(f) Show that if we know that $R(s) = 0$ for $s > k$, so that we only need to estimate $k+1$ values of $R(s)$ for $s = 0, \dots, k$, we get a good estimate of the variance of $\hat{\mu}_T$ by plugging in to the standard formula, in the sense that

$$T\big((X'X)^{-1}X'\Omega X(X'X)^{-1} - (X'X)^{-1}X'\hat{\Omega}_T X(X'X)^{-1}\big) \xrightarrow[T\to\infty]{P} 0\,,$$

while at the same time each of the two terms differenced in this expression converges in probability to a non-zero constant.