CONDITIONAL PDF, MARGINAL PDF, Γ EXERCISE

(1) Suppose the pdf of the wage W of a randomly selected fast-food worker is, conditional on the state minimum wage $\bar{W}$,

$$p(W \mid \bar{W}) = \begin{cases} e^{-(W-\bar{W})} & W \geq \bar{W} \\ 0 & W < \bar{W} \end{cases} .$$

(Note that, unlike in lecture, here we are not putting non-zero probability on $W = \bar{W}$.) The marginal distribution of $\bar{W}$ has pdf

$$q(\bar{W}) = \begin{cases} e^{-\bar{W}} & \bar{W} \geq 0 \\ 0 & \bar{W} < 0 \end{cases} .$$

(a) Find $E[w \mid \bar{W}]$, the marginal pdf $p(W)$ of w, $E[\bar{W} \mid W]$, and the conditional pdf $q(\bar{W} \mid W)$.

Letting $u = W - \bar{W}$, the conditional expectation is

$$\int_0^\infty (u + \bar{W}) e^{-u} \, du = 1 + \bar{W} .$$

(Using $\int we^{-w} dw = \Gamma(2) = 1$, $\int e^{-w} = 1$.)
The joint pdf is e^{-W} on the region $W > \bar{W} > 0$. The marginal for W is $\int_0^W e^{-W} \, d\bar{W} = We^{-W}$.
The conditional pdf of $\bar{W} \mid W$ is

$$\frac{e^{-W}}{\int_0^W e^{-W} \, d\bar{W}} = \frac{1}{W} \quad \text{on } (0, W),\ 0 \text{ elsewhere,}$$

i.e. uniform on $(0, W)$. Therefore $E[\bar{W} \mid W] = W/2$.

(b) Determine whether W and $\bar{W}$ are independent and explain how you reached your conclusion.

This is slightly tricky because the joint pdf is e^{-W} on the $W > \bar{W}$ region, and this is trivially a product of a function (e^{-W}) of W alone and a function (1) of $\bar{W}$ alone. But there is no way to define the two marginal pdf's so their product is zero over the $W < \bar{W}$ region, so the two variables are not independent. Of course this follows also, e.g., from the fact that $E[W \mid \bar{W}]$ depends on $\bar{W}$.

(2) The exponential pdf that showed up in problem 1 is a special case of the gamma distribution. A gamma random variable X is positive with probability one and has a pdf of the form

$$\gamma(x; p, \alpha) = \frac{\alpha^p x^{p-1} e^{-\alpha x}}{\Gamma(p)} .$$

Its expectation is p/α and its mode (maximum of the pdf) is at $x = (p-1)/\alpha$. The gamma function $\Gamma(p)$ is, like this distribution, well-defined for any $p > 0$ and has the properties $\Gamma(p) = (p-1)!$ for integer $p > 0$ and $p\Gamma(p) = \Gamma(p+1)$ for all $p > 0$. There is no formula for its values at non-integer p, but its values are tabulated in books and available with a function call in most computer languages.

Answer the same two questions as in problem 1, but with $\bar{W}$ distributed with the Γ distribution with $p = 2$, $\alpha = 1$ and with $\{W - \bar{W} \mid \bar{W}\}$ distributed as Γ with $p = 2$, $\alpha = 1$.

(a) *The joint pdf is $\bar{W}(W - \bar{W})e^{-W}$. The conditional expectation of $W \mid \bar{W}$ is $\bar{W} + 2$, from the cited results about means of Gamma distributions. The marginal pdf of W is $W^3 e^{-W}/6$, which is a Gamma(4,1) pdf.The conditional pdf of $\bar{W} \mid W$ is $6\bar{W}(W - \bar{W})/W^3$. This is a parabola with a peak at $\bar{W} = .5W$, which is also the conditional expectation. Note that this is can also be characterized as a Beta(2,2) distribution scaled to the $(0, W)$ interval. (We'll consider general Beta distributions later.)*

(b) *Here again W and $\bar{W}$ are not independent. This time the lack of independence appears even in the form of the joint pdf on the $W > \bar{W}$ region, which does not factor.*

(3) This is mental exercise. You'll get extra credit if you do it, but it's not required. Suppose the conditional distribution of $W \mid \bar{W}$ puts probability .5 on $W = \bar{W}$, with the remaining probability described by the density $.5e^{-(W-\bar{W})}$ on the $W > \bar{W}$ line segment. Suppose again that $\bar{W}$ has a gamma distribution with $p = 2$, $\alpha = 1$.

Answer the same questions as in problem 1, except that references to conditional and marginal density functions are replaced by references to conditional and marginal distributions, to allow for the possibility that there may be discrete components in the distributions.

$E[W \mid \bar{W}] = \bar{W} + .5$. This is calculated directly from the given conditional distribution. The marginal pdf of W is $g(W) = (.25W^2 + .5W)e^{-W}$. We calculate this by noting that the joint density over the $W > \bar{W}$ region is $.5\bar{W}e^{-W}$, so that if we integrate this over the rectangle with $\bar{W}$ ranging from 0 to W_0 and W ranging from W_0 to $W_0 + \delta$, we get approximately (for small δ) $.25\delta W_0^2 e^{-W_0}$. But there is an additional piece of probability along the $W = \bar{W}$ line within the rectangle, and it has probability approximately $.5\delta W_0 e^{-W_0}$. Dividing the sum of the two components by δ gives us the marginal density for W, evaluated at W_0.

To get the conditional distribution of $\bar{W} \mid W$, we consider the probability of the small rectangle with sides $[\bar{W}_0, \bar{W}_0 + \delta]$ and $[W_0, W_0 + \delta]$. If $W_0 \neq \bar{W}_0$ and δ is small enough, the probability of this rectangle will be approximately $\delta^2 \bar{W}_0 e^{-W_0}$. This is also, of course

$$E[\mathbf{1}_{\bar{W} \in [\bar{W}_0, \bar{W}_0 + \delta]} \cdot \mathbf{1}_{W \in [W_0, W_0 + \delta]}]$$
$$= E\left[E[\mathbf{1}_{\bar{W} \in [\bar{W}_0, \bar{W}_0 + \delta]} \mid W] \cdot \mathbf{1}_{W \in [W_0, W_0 + \delta]}\right] \doteq \delta^2 g(W_0) \cdot q(\bar{W}_0 \mid W_0)\,,$$

from the definition of conditional expectation and conditional density. So at a point like this ($W \neq \bar{W}$), the conditional distribution has a density, given by

$$\frac{.5\bar{W}e^{-W}}{g(W)} = \frac{\bar{W}}{.5W^2 + W} .$$

At a point where $W_0 = \bar{W}_0$, the probability of the small rectangle is approximately $.5\delta\bar{W}_0 e^{-W_0}$, which is, taking the expectation of the product of indicator functions as before, approximately equal to

$$\delta g(W_0) P[\bar{W} = W_0 \mid W_0] .$$

Thus

$$P[\bar{W} = W \mid W] = \frac{.5We^{-W}}{(.25W^2 + .5W)e^{-W}} = \frac{1}{.5W + 1} .$$

From these results it is straightforward to calculate that

$$E[\bar{W} \mid W] = W\frac{3 + W}{3 + 1.5W} \tag{1}$$

Of course for this problem, like the previous ones, W and $\bar{W}$ are not independent.

(4) Suppose X and Y are independent and both are uniformly distributed on $(-1, 1)$ (i.e. have a pdf that is .5 on that interval and 0 elsewhere). Suppose Z^* is uniformly distributed on $(0, 1)$ (i.e. has a pdf that is 1 on that interval and zero elsewhere), Z^* is independent of X and Y, and that $Z = \text{sign}(X \cdot Y)Z^*$. That is, $Z = Z^*$ when X and Y have the same sign, $Z = -Z^*$ when X and Y have different signs. Show that X, Y and Z all have the same marginal distribution, that they are pairwise independent, and that they are not mutually independent. What does the set on which they have positive joint density look like in 3d?

We already know that both X and Y are $U(-1, 1)$ (uniformly distributed on the interval (-1,1)). Since $P[XY > 0] = .5$ (because X and Y are independent and distributed symmetrically about zero), Z spreads half its probability on $(0, 1)$ and half on $(-1, 0)$, so Z also is $U(-1, 1)$. X and Y are pairwise independent. How about Y and Z? $P[XY > 0 \mid Y] = .5 = P[XY > 0]$. That is, knowledge of Y provides no information about the sign of XY, and therefore no information about the sign of Z. Thus $Z \mid Y$ is distributed $U(-1, 1)$, which is also its unconditional distribution. Therefore Y and Z are pairwise independent. Obviously this means, by symmetry, that X and Z are pairwise independent also. But the conditional distribution of $Z \mid X, Y$ is $U(0, 1)$ if $XY > 0$, $U(-1, 0)$ if $XY < 0$. Therefore, since the conditional distribution of $Z \mid X, Y$ does vary with X, Y, the three variables are not mutually independent. In X, Y, Z space the set on which the joint pdf has non-zero density consists of four cubes, linked at the edges. Their defining corners are (0,0,0) (which is for all four cubes one of the defining corners) and, respectively, (1,1,1), (-1,-1,1), (1,-1,-1), and (-1,1,-1).