CONDITIONAL PDF, MARGINAL PDF, Γ EXERCISE

(1) Suppose the pdf of the wage W of a randomly selected fast-food worker is, conditional on the state minimum wage $\bar{W}$,

$$p(W \mid \bar{W}) = \begin{cases} e^{-(W-\bar{W})} & W \geq \bar{W} \\ 0 & W < \bar{W} \end{cases}.$$

(Note that, unlike in lecture, here we are not putting non-zero probability on $W = \bar{W}$.) The marginal distribution of $\bar{W}$ has pdf

$$q(\bar{W}) = \begin{cases} e^{-\bar{W}} & \bar{W} \geq 0 \\ 0 & \bar{W} < 0 \end{cases}.$$

(a) Find $E[w \mid \bar{W}]$, the marginal pdf $p(W)$ of w, $E[\bar{W} \mid W]$, and the conditional pdf $q(\bar{W} \mid W)$.
(b) Determine whether W and $\bar{W}$ are independent and explain how you reached your conclusion.

(2) The exponential pdf that showed up in problem 1 is a special case of the gamma distribution. A gamma random variable X is positive with probability one and has a pdf of the form

$$\gamma(x; p, \alpha) = \frac{\alpha^p x^{p-1} e^{-\alpha x}}{\Gamma(p)}.$$

Its expectation is p/α and its mode (maximum of the pdf) is at $x = (p-1)/\alpha$. The gamma function $\Gamma(p)$ is, like this distribution, well-defined for any $p > 0$ and has the properties $\Gamma(p) = (p-1)!$ for integer $p > 0$ and $p\Gamma(p) = \Gamma(p+1)$ for all $p > 0$. There is no formula for its values at non-integer p, but its values are tabulated in books and available with a function call in most computer languages.

Answer the same two questions as in problem 1, but with $\bar{W}$ distributed with the Γ distribution with $p = 2$, $\alpha = 1$ and with $\{W - \bar{W} \mid \bar{W}\}$ distributed as Γ with $p = 2$, $\alpha = 1$.

(3) This is mental exercise. You'll get extra credit if you do it, but it's not required. Suppose the conditional distribution of $W \mid \bar{W}$ puts probability .5 on $W = \bar{W}$, with the remaining probability described by the density $.5e^{-(W-\bar{W})}$ on the $W > \bar{W}$ line segment. Suppose again that $\bar{W}$ has a gamma distribution with $p = 2$, $\alpha = 1$.

Answer the same questions as in problem 1, except that references to conditional and marginal density functions are replaced by references to conditional and marginal distributions, to allow for the possibility that there may be discrete components in the distributions.

(4) Suppose X and Y are independent and both are uniformly distributed on $(-1,1)$ (i.e. have a pdf that is .5 on that interval and 0 elsewhere). Suppose Z^* is uniformly distributed on $(0,1)$ (i.e. has a pdf that is 1 on that interval and zero elsewhere), Z^* is independent of X and Y, and that $Z = \text{sign}(X \cdot Y)Z^*$. That is, $Z = Z^*$ when X and Y have the same sign, $Z = -Z^*$ when X and Y have different signs. Show that X, Y and Z all have the same marginal distribution, that they are pairwise independent, and that they are not mutually independent. What does the set on which they have positive joint density look like in 3d?