Problem Set 2

QUESTION 1:

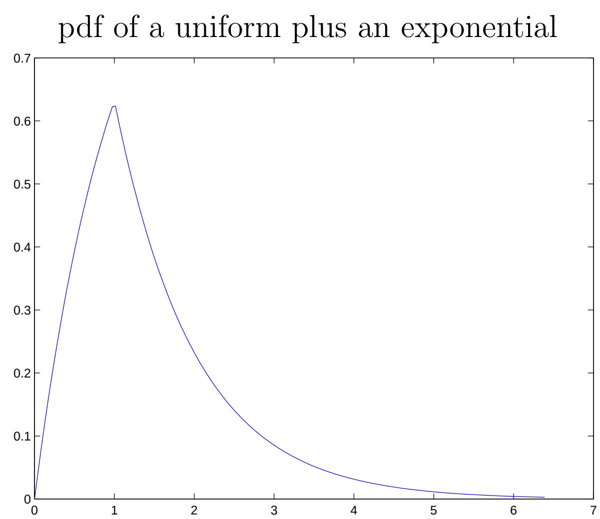
As seen in class, the density of $U = X + Y$ is given by $f_U(u) = f_X \star f_Y(u) = \int f_X(u-v) f_Y(v) dv$. As indicated in the exercise, $f_X = 1_{[0,1]}$ and $f_Y = e^{-v} 1_{[o,\infty)}$. So,

$$f_U(u) = \int 1_{[0,1]}(u-v) \times e^{-v} 1_{[o,\infty)}(v) dv$$

so the relevant sets are $0 \leq u - v \leq 1, v \geq 0$. This breaks down in two cases: 1. if $u \geq 1$, in which case $u - 1 \leq v \leq u$ and 2. if $u \in [0,1)$, in which case $0 \leq v \leq u$. So,

$$f_U(u) = \begin{cases} \int_{u-1}^{u} e^{-v} dv = e^{-u}(e-1) & \text{if } u \geq 1 \\ \int_0^u e^{-v} dv = 1 - e^{-u} & \text{if } u \in [0,1]. \\ 0 & \text{otherwise.} \end{cases}$$

Graphically the density function looks like:



QUESTION 2:

1. $\min\{X, Y\} = Z$.

The cdf's of the two variables are

$$F_X(a) = \begin{cases} 0 & a < 0 \\ a & a \in [0,1] \\ 1 & a > 1 \end{cases}$$

$$F_Y(a) = \begin{cases} 1 - e^{-a} & a > 0 \\ 0 & a \leq 0 \end{cases}.$$

Because they are independent, their joint cdf is the product of these two, and the pdf of their maximum, which we observed in class to be $F_{\max}(a) = F_{XY}(a,a)$, where

F_{XY} is the joint cdf, is therefore

$$F_X(a)F_Y(a) = \begin{cases} 0 & a < 0 \\ a \cdot (1 - e^{-a}) & a \in [0,1] \\ 1 - e^{-a} & a > 1 \end{cases} .$$

The pdf is then just the derivative of this one-dimensional cdf, which is

$$f_{\max}(a) = f_X(a)F_Y(a) + F_X(a)f_Y(a) = \begin{cases} 0 & a < 0 \\ 1 - (1-a)e^{-a} & a \in [0,1] \\ e^{-a} & a > 1 \end{cases} .$$

For the minimum, the cdf is $1 - (1 - F_X(a))(1 - F_Y(a))$, so the pdf emerges as

$$f_{\min}(a) = f_X(a)(1 - F_Y(a)) + (1 - F_X(a))f_y(a) = f_y(a) + f_x(a) - f_{\max}(a)$$
$$= \begin{cases} 0 & a < 0 \\ 1 + e^{-a} - 1 + (1-a)e^{-a} = (2-a)e^{-a} & a \in [0,1] \\ 0 & a > 1 \end{cases} .$$

QUESTION 3:

Because X^2 is a non-negative random variable, it suffices to show that, if Y is a non-negative random variable, it is not possible to have b such that

$$(\forall a > b)P(\{Y \geq a\}) = \frac{EY}{a}$$

If it were, the df for Y on $[b, \infty)$ could be seen to be:

$$F_Y(a) = 1 - P(y \geq a) = 1 - \frac{EY}{a} \text{ (if } a > b)$$
$$\Rightarrow$$
$$EY = E(Y1_{y \leq b}) + E(Y1_{y > b}) \geq$$
$$\geq E(Y1_{y>b}) = \int 1_{a \geq b} a dF_Y(a) =$$
$$= \int_b^\infty a\frac{EY}{a^2}da = EY \times [\lim_{a\to\infty} \log a - \log b] = \infty ,$$

which contradicts the assumption, required to apply the inequality, that $EY < \infty$.

QUESTION 4

I didn't think through what a mess it was going to be to get normalizing constants for these pdf's. If it were impossibly difficult, it wouldn't have been so bad, but it's actually possible, with more work than I intended to give you.

The easy part of the answer is

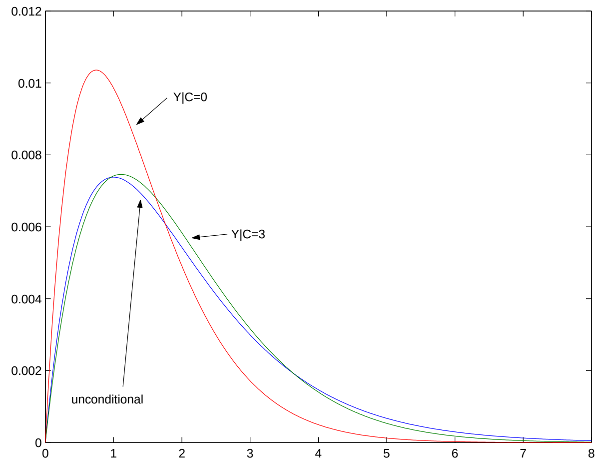
(a) Because the joint pdf for $C_i > 0$ is $ye^y\varphi(c - \gamma - y\beta; \sigma^2)$, the conditional pdf is *proportional* to this joint pdf, treated as a function of y, i.e. proportional to

$$ye^{-y-\frac{1}{2}(c-\gamma-y\beta)^2/\sigma^2} .$$

(b) Because the base measure for the conditional distribution of $C_i \mid Y_i$ does not vary with Y_i, the we can treat $P[C_i = 0 \mid Y_i]$ as a density with respect to this base measure, and then apply the usual Bayes' rule formula. This means that the conditional density of $Y_i \mid \{C_i = 0\}$ is *proportional* to $ye^{-y}\Phi(-(\gamma + y\beta)/\sigma)$.

(c) With a computer, it's easy to find normalizing factors numerically and plot the pdf's.

Tobit pdf's for Y



Note that seeing $C_i = 0$ moves your beliefs about Y_i substantially away from the unconditional distribution, whereas $C_i = 3$, since that is very near the unconditional expected value of C_i, $\gamma + EY_i\beta = 2.74$, hardly changes your conditional distribution for Y_i at all. (What is the scale to use in deciding whether 2.74 is "very near" 3 in this context?)

The nasty part of the answer is finding explicit normalizing factors. This is possible, using a few tricks. The tricks are:

(i) When a pdf is proportional to $e^{p(x)}$, where $p(x)$ is any quadratic function of x with a negative coefficient on the x^2 term, then it is proportional to a normal pdf with some mean and variance. This means that it is possible to find its definite integral. To find the implied mean and variance, and thus the normalizing constant, requires "completing the square". We'll go through this in class before long, though since the main idea is usually taught in high school algebra, it was possible for you to figure it out for yourself, in principle.

(ii) When a pdf is proportional to $yf(y)$, where $f(y)$ is a distribution with known mean, the normalizing constant can be computed because the integral of $yf(y)$ will be the expectation of the distribution determined by the density f.

(iii) Integrating the conditional pdf given $C_i = 0$, since it involves Φ, looks like a mess. But it turns out that it can be converted to the same kind of integral as in the $C_i = 3$ case, using integration by parts.

If it turns out that some people put a lot of effort into getting these integrals analytically, I'll post explicit answers.