SNLM EXERCISE

Suppose you are estimating a demand curve

$$x(t) = \gamma + \beta y(t) + \theta p(t) + \varepsilon(t)\,,$$

where x is the quantity purchased, y is income, and p is the price. All data have been measured in natural logarithms. Letting

$$\underset{T\times 1}{X} = \begin{bmatrix} x(1) \\ \vdots \\ x(T) \end{bmatrix}, \quad \underset{T\times 3}{Z} = \begin{bmatrix} 1 & y(1) & p(1) \\ \vdots & \vdots & \vdots \\ 1 & y(T) & p(T) \end{bmatrix}, \quad \underset{T\times 1}{U} = \begin{bmatrix} \varepsilon(1) \\ \vdots \\ \varepsilon(T) \end{bmatrix},$$

you are willing (perhaps mistakenly) to assume that $U \mid X \sim N(0, \sigma^2 I)$. Because suppliers with pricing power charge more in high-income areas, y and p have a strong positive sample correlation. The sample moment matrices are

$$Z'Z = \begin{bmatrix} 50 & 100 & 100 \\ 100 & 250 & 240 \\ 100 & 240 & 250 \end{bmatrix}, \quad Z'X = \begin{bmatrix} 180 \\ 424 \\ 413 \end{bmatrix}, \quad X'X = 780\,.$$

(a) With a flat prior on β (i.e. treating the likelihood as proportional to the posterior pdf), find the posterior mean vector and variance matrix for the coefficient vector (γ, β, θ). [Note: the variance of a multivariate $t_n(0,\Sigma)$ distribution is $(n/(n-2))\Sigma$. Observe that you can tell from the the upper left element of $Z'Z$ that the sample size is $T = 50$.]

(b) Plot level curves of the posterior distribution of β, θ. [Note that if $\xi \sim t_n(0,\Sigma)$, every subvector of ξ also has a t_n distribution, with scale matrix given by the corresponding submatrix of Σ. As a reminder, the pdf of a multivariate $\xi \sim t_n(0,\Sigma)$ is

$$\frac{\kappa |\Sigma|^{-1/2}}{\left(1 + \xi'\Sigma^{-1}\xi/n\right)^{(n+k)/2}}\,,$$

where κ is a scale factor that makes the pdf integrate to one (not needed for this exercise) and k is the dimension of the ξ vector.

(c) Now suppose that you need to use this result for actual decision-making, and you want to take account of your prior belief that a price elasticity is likely to be negative and income elasticity positive. So you redo your analysis using, instead of a flat prior on σ^2 and (γ, β, θ), a prior pdf proportional to

$$\sigma^{-3} \exp\left(-\frac{1}{2}\frac{9 * (\psi - \bar{\psi})'(\psi - \bar{\psi})}{\sigma^2}\right),$$

where $\bar{\psi} = (1; -1)'$ is your prior mean for $\psi = (\beta, \theta)'$. This is a conjugate prior (though, because it does not involve γ, it is still flat in γ and thus not a proper prior.) Find the new posterior mean and variance matrix.

(d) Show on the same graph the contours in (β, θ) of the original posterior (under the flat prior) and the contours of the prior, marking the points corresponding to the estimate of ψ with and without the prior. [The prior's contours can be shown for some fixed value of σ^2. They will have the same shape for any value of σ^2, and also would have the same shape if σ^2 were integrated out.]

[To make a contour plot of a function on $\mathbb{R}^2$ with matlab, use `meshgrid` to create 2D matrices filled with appropriate values of the two argument variables, then compute the levels of the function over the grid using a matlab formula, then apply the `contour` command. To mark a point, `hold` the current graph, and then `plot(x,y,'+')` (where the character in quotes has several possible values).]