ANSWER TO FINAL EXAM PROBLEM 1

(1) (35 points) Under model A the random variable X has cdf

$$F_A(x) = \begin{cases} 0 & x < 0 \\ \frac{8}{3}x - \frac{16}{9}x^2 & x \in [0, .75] \\ 1 & x > .75 \end{cases} .$$

Under model B the random variable X has the cdf

$$F_B(x) = \begin{cases} 0 & x < .5 \\ 4x^2 - 4x + 1 & x \in [.5, 1] \\ 1 & x > 1 \end{cases} .$$

(a) Construct a test of the null hypothesis that model B is true, making the rejection region have the form of an interval $(0, a)$. Calculate a so that the test has a significance level of .04.

(10 points) We just need to find the point a that solves $F_B(a) = .04$, i.e. $(2x-1)^2 = .04$ This equation has two roots, but one of them is below .5 and thus in the region where $F_A = 0$, not where this formula applies for the cdf. The other root is $x = .6$.

(b) Calculate the power of the test against model A as the alternative.

(10 points) This is just

$$F_A(.6) = \frac{8 \cdot .6}{3} - \frac{16 \cdot .36}{9} = 1.6 - .64 = .96 .$$

(c) Is the test biased? Explain why or why not.

(10 points) It's not biased, because the power exceeds the significance level. For a simple null against a simple alternative, there is nothing more to biasedness of a test than this.

(d) Suppose the observed value of X is a, or just slightly below a. Show that though this leads to rejection of model B using your proposed test, this observation actually constitutes evidence in *favor* of model B, in the sense that model B is more likely, conditional on this observed value of X, than it was before X was observed.

(15 points) The probability of model B given the observation $X = .6$ is

$$\frac{\pi(B) f_B(.6)}{\pi(B) f_B(.6) + \pi(A) f_A(.6)} ,$$

where f_B and f_A are the density functions for X under the two models and $\pi(A), \pi(B)$ are the prior probabilities of the two models, *before the observation is made. The formula is a standard application of Bayes' rule. The joint probability density over the models and the observations is $\pi(A)f_A(x)$ where the model is A and $\pi_B f_B(x)$ over the part of the space where the model is B. The posterior probability is then the joint pdf divided by the marginal pdf evaluated at the observed X.*

Since $f_A(x) = \frac{8}{3} - \frac{32}{9}x$ and $f_B(x) = 8x - 4$, we can see that at $x = .6$ the ratio of likelihoods is $f_B(.6)/f_A(.6) = .8/.533 = 1.5$, so the posterior probability of model B will be higher than its prior probability. For example, if the prior was $\pi(A) = \pi(B) = .5$, then the posterior probability of model B would be .6. Graphically, the situation is as below: