EXERCISE ON INDEPENDENCE, SUMS, MOMENTS

(1) X is distributed as $U(0,1)$, i.e. it has pdf equal to 1 on $[0,1]$ and zero elsewhere. Y is independent of X and distributed with pdf e^{-y} on $[0,\infty)$, zero elsewhere. What is the pdf of $X+Y$? Show it analytically and also sketch it.

(2) For this same X and Y, what are the pdf's of $\min\{X,Y\}$ and $\max\{X,Y\}$?

(3) Prove that there is no distribution on $\mathbb{R}^1$ for which the the Tchebychev inequality holds with equality everywhere on a set of the form $|x|>b$. That is, there is no pair of random variable X and real number b such that

$$(\forall a>b)P[|X|\geq a]=\frac{EX^2}{a^2}\,.$$

(4) A classic econometric model is the Tobit model. We observe (say) consumption C_i and income Y_i for individuals $i=1,\dots,n$, and we believe the conditional distribution of $C_i \mid Y_i$ has the $N(\gamma+Y_i\beta,\sigma^2)$ pdf over the set $C_i>0$, and a discrete lump of probability equal to $\Phi(-(\gamma+Y_i\beta)/\sigma)$, where Φ is the cdf of a $N(0,1)$ distribution, at $C_i=0$. The motivation here is that $C_i \mid Y_i$ is thought of as generated by first drawing from a normal distribution, then converting any negative values to zero before reporting them.

Suppose we know that $\sigma^2=9$, $\gamma=1$ and $\beta=.87$. We also know that the marginal distribution of income Y_i has pdf ye^{-y} on $[0,\infty)$.

(a) What is the conditional distribution of $Y_i \mid \{C_i=3\}$?

(b) What is the conditional distribution of $Y_i \mid \{C_i=0\}$? *In each of these cases Y has a conditional density that can be written out. You can use the standard normal cdf Φ in the expression.*

(c) Sketch or draw with a computer, on the same graph, both conditional pdf's for Y_i and the marginal pdf ye^{-y}. Be sure you have normalized so all three have the same integral.