Problem Set 1

Question 1:

(a) $F(x,y) = P(\{X \leq x, Y \leq y\}) = P(\{(x,y) \in (-\infty, x] \times (-\infty, y]\})$

Case 1: If $x \geq 1, y \geq 1$, then $(-\infty, 1]^2 \subset (-\infty, x] \times (-\infty, y]$. In this case, $P(\{(x,y) \in (-\infty, 1]^2\}) = F(1,1) = 1 \Rightarrow F(x,y) = 1$.

Case 2: If $x \leq 0, y \leq 0$, then $(-\infty, x] \times (-\infty, y] \supset (-\infty, x] \times (-\infty, y]$. In this case, $P(\{(x,y) \in (-\infty, 0]^2\}) = F(0,0) = 0 \Rightarrow F(x,y) = 0$.

Case 3: If $x \leq 0, y \in [0,1]$, then $(-\infty, 0]^2 \subset (-\infty, 0] \times (-\infty, y]$. In this case, $0 \leq F(x,y) \leq F(0,y) = 0 \Rightarrow F(x,y) = 0$.

Case 4: If $x \in [0,1], y \leq 0$, the analysis is similar to the previous case and $F(x,y) = 0$.

Case 5: If $x \leq 1, y \geq 1$, then, since $\Re^2 \setminus (-\infty, 1]^2 = 1 - F(1,1) = 0, P(\{(x,y) \in A \subset \Re^2 \setminus (-\infty, 1]^2\}) = 0 \Rightarrow F(x,y) - F(x,1) = 0 \Rightarrow F(x,y) = x$ if $x \in [0,1]$ (since $F(x,1) = x$ in this case) and $F(x,y) = 0$ if $x \leq 0$ (since $F(x,1) = 0$ in this case).

Case 6: If $x \geq 1, y \leq 1$, the analysis is similar to the previous case.

(b) Using the definition introduced in class:

$$F_X(x) = \lim_{y \to \infty} F(x,y) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \in (0,1) \\ 1 & \text{if } x \geq 1. \end{cases}$$

and

$$F_Y(y) = \lim_{x \to \infty} F(x,y) = \begin{cases} 0 & \text{if } y \leq 0 \\ y & \text{if } y \in (0,1) \\ 1 & \text{if } y \geq 1. \end{cases}$$

(c) $U = lnX, V = lnY \iff X = e^U, Y = e^V$. So,

$$G(u,v) = \begin{cases} \frac{1}{2}(e^{u+v} + e^{min\{u,v\}}) & \text{if } (u,v) \in (-\infty, 0]^2 \\ 1 & \text{if } (u,v) \in (0, \infty)^2 \\ e^v & \text{if } (u,v) \in (0,\infty) \times (-\infty, 0) \\ e^u & \text{if } (u,v) \in (-\infty, 0) \times (0, \infty) \end{cases}$$

(d) If the joint distribution of X, Y had a density with respect to the Lebesgue measure it would coincide with $\partial^2 F(\cdot,\cdot)/\partial x \partial y$ a.e.. In fact,

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} \begin{cases} = \frac{1}{2} & \text{on the unit square (except at the } x = y \text{ locus)} \\ = 0 & \text{otherwise.} \end{cases}$$

But $\int \partial^2 F/\partial x \partial y d\lambda$ (where λ denotes the Lebesgue measure) does not recover $F(\cdot,\cdot)$. The reason for this is that $F(\cdot,\cdot)$ represents a measure that concentrates weight along the diagonal line and as a consequence is not absolutely continuous with respect to the Lebesgue measure.

The marginal distributions are uniform and consequently have densities with respect to the Lebesgue measure (on the line).

The marginal distributions of U and V are given by the same function:

$$G_U(u) = \lim_{v \to \infty} G(u,v) = \begin{cases} e^u & \text{if } u < 0 \\ 1 & \text{if } u \leq 0. \end{cases}$$

This function is easily seen to have a density equal a.e. to:

$$g_U(u) = \lim_{v \to \infty} G(u,v) = \begin{cases} e^u & \text{if } u \leq 0 \\ 0 & \text{if } u > 0. \end{cases}$$

(e) To obtain the conditional expectation of $Y|X$ one should notice that the joint distribution of X and Y is a mixture of two distributions. The first is the uniform distribution on the unit square, whose distribution and density with respect to the Lebesgue measure (on $\Re^2$) are given by:

$$F(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (1,\infty)^2 \\ xy & \text{if } (x,y) \in [0,1]^2 \\ x & \text{if } (x,y) \in [0,1] \times [1,\infty) \\ y & \text{if } (x,y) \in [1,\infty) \times [0,1] \\ 0 & \text{if } (x,y) \in \Re^2 \setminus (0,\infty) \end{cases}$$

and the density is a.e. equal to

$$f(x,y) = \begin{cases} 1 & \text{if } (x,y) \in [0,1]^2 \\ 0 & \text{otherwise} \end{cases}$$

The second distribution portrays a random vector which is uniformly distributed on the locus $x = y, (x,y) \in [0,1]^2$. The distribution function for this random vector is given by:

$$F(x,y) = \begin{cases} 1 & \text{if } (x,y) \in (1,\infty)^2 \\ min\{x,y\} & \text{if } (x,y) \in [0,1]^2 \\ x & \text{if } (x,y) \in [0,1] \times [1,\infty) \\ y & \text{if } (x,y) \in [1,\infty) \times [0,1] \\ 0 & \text{if } (x,y) \in \Re^2 \setminus (0,\infty)^2 \end{cases}$$

This distribution has no density with respect to the Lebesgue measure (on $\Re^2$) because it assigns positive probability to a subset of the diagonal line $x = y$, whose measure is null under the Lebesgue measure. It nonetheless has a density with respect to the (rotated) Lebesgue measure on the diagonal which assigns $1/\sqrt{2}$ if $x = y, (x,y) \in [0,1]^2$.

It can be easily seen from the joint distribution of X and Y that it mixes these two distributions with equal probability. We can work out the conditional expectation separating these two cases. The conditional expectation of Y for the first distribution is $1/2$ irrespective of X. For the second case, $X = x$ implies that $Y = x$. So, the conditional expectation of $Y|X$ is given by

$$E(Y|X = x) = .5 \times .5 + .5 \times x = .25 + .5 \times x.$$

For $U|V$ (and $V|U$ symmetrically), an analogous reasoning allows one to derive:

$$E(U|V = v) = .5 \times -1 + .5 \times v = -.5 + .5 \times v.$$

Question 2:

(a) The required density can be obtained through the Jacobian method:

$$\begin{aligned} g(\rho, \theta) &= f(x(\rho, \theta), y(\rho, \theta))|det^{-1}\left(\tfrac{\partial(\rho,\theta)}{\partial(x,y)}\right)| = \\ &= \tfrac{1}{2\pi} exp(-\rho^2/2)|det^{-1}\left(\tfrac{\partial(\rho,\theta)}{\partial(x,y)}\right)| \end{aligned}$$

where $(\rho, \theta) = (\sqrt{x^2 + y^2}, arctan(y/x))$. So

$$det\left(\frac{\partial(\rho, \theta)}{\partial(x, y)}\right) = \begin{bmatrix} x/\sqrt{x^2+y^2} & y/\sqrt{x^2+y^2} \\ -y/(x^2+y^2) & x/(x^2+y^2) \end{bmatrix} = \frac{1}{\rho}.$$

Given this, the expression is then:

$$g(\rho, \theta) = \begin{cases} \frac{1}{2\pi} exp(-\rho^2/2)\rho & \text{if } \rho > 0, \theta \in [-\pi, pi] \\ 0 & \text{otherwise} \end{cases}$$

(b) Integrating the above one gets the following expression:

$$G(\rho, \theta) = \begin{cases} 0 & \text{if } \rho \leq 0 \text{ or } \theta \leq -\pi \\ (1/2\pi - e^{-\rho^2/2}/2\pi)(\theta + \pi) & \text{if } \rho \geq 0 \text{ and } \theta \in [-\pi, \pi] \\ 2\pi(1/(2\pi) - e^{-\rho^2/2}/(2\pi)) & \text{otherwise} \end{cases}$$

(c) The required expression is given by:

$$p(\rho|\theta = \pi/2) = \begin{cases} \frac{exp(-\rho^2/2)\rho/(2\pi)}{\int_0^\infty exp(-\rho^2/2)\rho/(2\pi) d\rho} & \text{if } \rho > 0 \\ 0 & \text{otherwise} \end{cases}$$

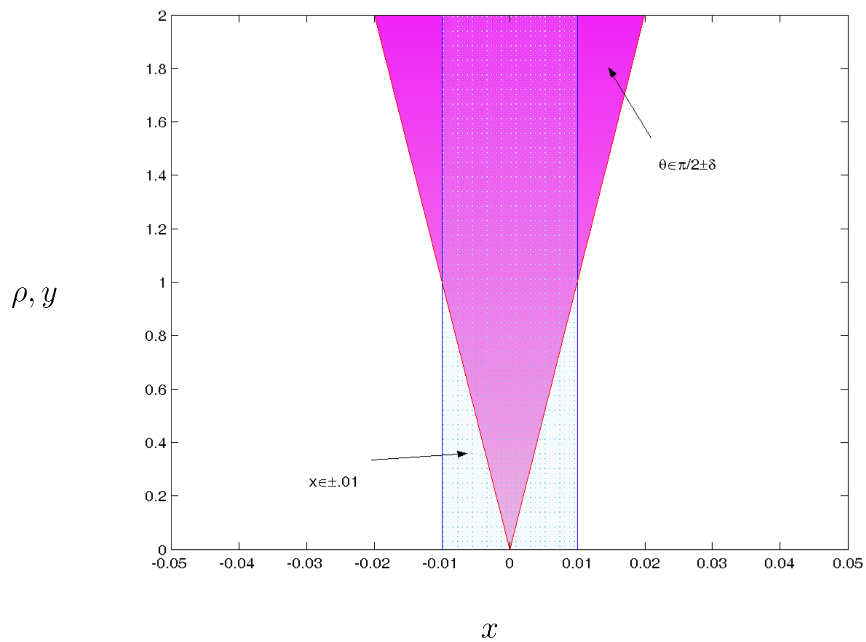
The integral in the denominator can be seen to be $(2\pi)^{-1}$ so that

$$p(\rho|\theta = \pi/2) = \begin{cases} exp(-\rho^2/2)\rho & \text{if } \rho > 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) Though the problem didn't say so, you were meant to assume here that the distribution of X, Y is $N(0, I)$, not a general joint normal, so that this is just the distribution of the parts of the question above, restricted to the $y > 0$ region (i.e., to $\Re \times \Re^+$). It will therefore be the original $N(0, I)$ pdf, multiplied by 2 so that it integrates to 1 over its domain. The conditional pdf here is then the joint pdf evaluated at $x = 0$, scaled to integrate to 1 over y. That is,

$$p(y \mid x) = \frac{\frac{1}{2\pi} 2e^{-\frac{1}{2}y^2}}{\int_0^\infty \frac{1}{2\pi} 2e^{-\frac{1}{2}y^2} dy} = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}y^2} \; . \tag{1}$$

Even though the set $\{(x, y) \mid x = 0, y > 0\}$ is the same set of points in $\Re^2$ as the set $\{(x, y) \mid \theta(x, y) = \pi/2\}$, and within this set $y = \rho(x, y) = \rho(0, y) = \sqrt{0 + y^2}$, the answer to this problem shows that $p(y \mid x = 0) \neq p(\rho \mid \theta = 0)$. What's the intuition? Remember that conditioning on values of a continuously distributed random variable is not the same thing as conditioning on the (zero-probability) set where the variable takes on a particular value. One can think of the distribution of $\rho \mid \{\theta = \theta_0\}$ as the limit as $\delta \to 0$ of the distribution of $\rho \mid \{\theta \in (\theta_0 \pm \delta)\}$ and of the distribution of $Y \mid \{X = x_0\}$ as the limit of the distribution of $Y \mid \{X \in (x_0 \pm \delta)\}$. The graph shows the nature of the two δ-sets. The pie-shaped $\theta = \pi/2 \pm \delta$ set has very little probability near $\rho = 0$, compared to the probability near $y = 0$ in the $x = 0 \pm \delta$ set, because the pie gets very narrow there relative to the rectangular region.



Note that the level curves of a $N(0, I)$ pdf would show on this graph as almost perfectly straight horizontal lines.