Econ. 517, Econometrics I, part I   Fall 2001   Chris Sims

Exercise on Probability and Expectation: Answers*

Apparently no one had major difficulties with problems 1 or 2, so there are no written-out answers for those here.

3. (a) Suppose that the random variable X on $\mathcal{S}$ is defined by

$$X(1) = 5 \quad X(2) = 4 \quad X(3) = 3 \quad X(4) = 2 \quad X(5) = 1\,. \tag{1}$$

For each of the $\mathcal{G}_i$ of problem 1, using a probability P on $\mathcal{S}$ that you found in problem 2 to be internally consistent, find $E[X \mid \mathcal{F}_i]$, where $\mathcal{F}_i$ is the σ-field generated by $\mathcal{G}_i$. (This conditional expectation is a random variable on $\mathcal{S}$, of course, so it is a list of 5 numbers, the value of the random variable at each point in $\mathcal{S}$.)

(b) Suppose that the random variable Z on $\mathcal{S}$ is defined by

$$Z(1) = 2 \quad Z(2) = 2 \quad Z(3) = 3 \quad Z(4) = 3 \quad Z(5) = 5\,. \tag{2}$$

Show that $E[X \mid Z]$ coincides with one of the $E[X \mid \mathcal{F}_i]$ random variables that you computed in part 3a.

(a) For a finite discrete $\mathcal{S}$ like this, a σ-field can be characterized by a list of non-intersecting subsets such that all non-null sets in the σ-field are unions of subsets in the list. For $\mathcal{G}_1$, the list is just the three sets that are given in the problem statement. A function is measurable with respect to such a discrete-space sigma field if an only if it is constant on each set in the list. And for any A in the list, $E[X \mid \mathcal{G}_1]$ is equal to $E[X \mid A]$ at points in A. The internally consistent probability from problem 2 is the second, and it puts probabilities 1/12, 1/6, 1/4, 1/4, 1/4 on the five points 1, 2, 3, 4, 5. So the conditional expectation we are looking for has the values on these five points 13/3, 13/3, 2.5, 2.5, 1. It is not hard to check that this is $\mathcal{F}_1$-measurable, and that it satisfies the defining properties of a conditional expectation. For $\mathcal{G}_2$, the σ-field generated includes all subsets of $\mathcal{S}$, so $E[X \mid \mathcal{F}_2] = X$.

$\mathcal{G}_3$ is itself a σ-field, and there are two non-intersecting sets such that all non-null elements of $\mathcal{G}_3$ are unions of the 2: the even and the odd numbers in $\{1,2,3,4,5\}$. The conditional expectation of X, then, is $2\frac{3}{7}, 2.8, 2\frac{3}{7}, 2.8, 2\frac{3}{7}$. To get the $2\frac{3}{7}$, for example, we calculate

$$\frac{5 \cdot \frac{1}{12} + 3 \cdot \frac{3}{12} + 1 \cdot \frac{3}{12}}{\frac{1}{12} + \frac{3}{12} + \frac{3}{12}} = \frac{17}{7}.$$

(b) It is easy to see that Z is constant within each of the three non-intersecting sets in $\mathcal{G}_1$ whose unions make up $\mathcal{F}_1$, so it is $\mathcal{F}_1$-measurable. In fact also each of the sets in $\mathcal{G}_1$ can be characterized as $\{x \in \mathcal{S} \mid Z(x) = n\}$ for some n, so Z generates the $\mathcal{F}_1$ σ-field. So as a random variable on $\mathcal{S}$, $E[X \mid Z] = E[X \mid \mathcal{G}_1]$.

4. Consider the functions $F : \mathbb{R}^k \to [0,1]$ defined by

   (a)
$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ .1 + .2x \cdot y & x > 0,\ y > 0,\ x \cdot y < 4.5 \\ 1 & x \cdot y \geq 4.5 \end{cases}$$

   (b)
$$F(x,y) = \frac{4}{\pi^2} \arctan(e^x) \arctan(e^y)$$

   (c)
$$F(x,y,z) = \begin{cases} \min\{x,1\} \cdot \min\{y,1\} \cdot \min\{z,1\} & \text{if } x > 0,\ y > 0, \text{ and } z > 0 \\ 0 & \text{if } x < 0 \text{ or } y < 0 \text{ or } z < 0 \end{cases}$$

   Note that this means that

$$F(x,y,z) = xyz \qquad \text{if } x \in (0,1) \text{ and } y \in (0,1) \text{ and } z \in (0,1)$$

   In each case, determine whether F is a distribution function. If so, in cases 4a and 4b find the probability of the rectangle with corners (-1,-1) and (3,3), and in case 4c find the probability of a sphere of radius .5 centered at (.5,.5,.5).

(a) This isn't a distribution function because, e.g., it implies that the probability of the rectangle with corners at (1,1) and (3,3) is 1+.3-.7-.7=-.1<0. If you didn't notice this, you might have computed using the correct formula that the probability of the rectangle with corners at (-1,-1) and (3,3) is 1+0-0-0=1. Note that the discontinuity at points where $x = 0$ or $y = 0$ is OK. cdf's can be discontinuous, so long as they are *right*-continuous. It was also no problem that you were not told the value of F at points where x or y is exactly zero. If F was to be a distribution function, its values at these points had to be right-limits, i.e. the limit of $.1 + .2xy$ as x or y approaches zero from above. This is always .1 .

(b) This is a distribution function, because it is continuous everywhere, goes to zero at $-\infty$ and 1 at ∞, and it has

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{1}{(e^{-x} + e^{x})(e^{-y} + e^{y})}$$

For this F, which implies independence, the probability of the $[(-1, -1), (3, 3)]$ box is

$$\frac{4\left(\arctan(e^3) - \arctan(e^{-1})\right)^2}{\pi^2} = .5534 \,.$$

(c) This is a distribution function, and it describes a uniform distribution over the unit cube, i.e. a pdf equal to 1 everywhere inside the cube with corners (0,0,0) and (1,1,1), and zero everywhere else. Note that this F is also a product, and thus describes three random variables that are independent. The probability of the sphere centered at (.5,.5,.5) of radius .5 is, since it lies entirely inside the cube, just the volume of the sphere, i.e., as you no doubt recall from high school, $(4/3)\pi r^3 = .5236$.