Econ. 517, Econometrics I, part I — Fall 2001 — Chris Sims

Exercise on Probability and Expectation*

1. For each of the classes of sets $\mathcal{G}_i$ below, Show that $\mathcal{G}_i$ is, or is not, itself a σ-field. If it is not, display the σ-field generated by $\mathcal{G}_i$. In each case, the underlying complete space $S = \{1,2,3,4,5\}$.

$$\mathcal{G}_1 = \{\{1,2\},\{3,4\},\{5\}\} \tag{1}$$
$$\mathcal{G}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\}\} \tag{2}$$
$$\mathcal{G}_3 = \{\{\},\{1,3,5\},\{2,4\},\{1,2,3,4,5\}\} \tag{3}$$

2. Suppose we define the following function on $\mathcal{G}_2$:

$$P[\{1,2\}] = .1 \quad P[\{2,3\}] = .2 \quad P[\{3,4\}] = .4 \quad P[\{4,5\}] = .5\,. \tag{4}$$

   (a) Is there a probability defined on the σ-field generated by $\mathcal{G}_2$ that matches this function on these sets? Why or why not? If so, are the probabilities of the 5 individual points in $\mathcal{S}$ uniquely determined? If so, what are they?

   (b) Answer the question again, this time with

$$P[\{1,2\}] = \frac{1}{4} \quad P[\{2,3\}] = \frac{5}{12} \quad P[\{3,4\}] = \frac{1}{2} \quad P[\{4,5\}] = \frac{1}{2}\,. \tag{5}$$

3. (a) Suppose that the random variable X on $\mathcal{S}$ is defined by

$$X(1) = 5 \quad X(2) = 4 \quad X(3) = 3 \quad X(4) = 2 \quad X(5) = 1\,. \tag{6}$$

   For each of the $\mathcal{G}_i$ of problem 1, using a probability P on $\mathcal{S}$ that you found in problem 2 to be internally consistent, find $E[X \mid \mathcal{F}_i]$, where $\mathcal{F}_i$ is the σ-field generated by $\mathcal{G}_i$. (This conditional expectation is a random variable on $\mathcal{S}$, of course, so it is a list of 5 numbers, the value of the random variable at each point in $\mathcal{S}$.)

(b) Suppose that the random variable Z on $\mathcal{S}$ is defined by

$$Z(1) = 2 \quad Z(2) = 2 \quad Z(3) = 3 \quad Z(4) = 3 \quad Z(5) = 5\,. \tag{7}$$

Show that $E[X\,|\,Z]$ coincides with one of the $E[X\,|\,\mathcal{F}_i]$ random variables that you computed in part 3a.

4. Consider the functions $F : \mathbb{R}^k \to [0,1]$ defined by

(a)
$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ .1 + .2x \cdot y & x > 0,\ y > 0,\ x \cdot y < 4.5 \\ 1 & x \cdot y \geq 4.5 \end{cases}$$

(b)
$$F(x,y) = \frac{4}{\pi^2} \arctan(e^x) \arctan(e^y)$$

(c)
$$F(x,y,z) = \begin{cases} \min\{x,1\} \cdot \min\{y,1\} \cdot \min\{z,1\} & \text{if } x > 0,\ y > 0, \text{ and } z > 0 \\ 0 & \text{if } x < 0 \text{ or } y < 0 \text{ or } z < 0 \end{cases}$$

Note that this means that

$$F(x,y,z) = xyz \qquad \text{if } x \in (0,1) \text{ and } y \in (0,1) \text{ and } z \in (0,1)$$

In each case, determine whether F is a distribution function. If so, in cases 4a and 4b find the probability of the rectangle with corners (-1,-1) and (3,3), and in case 4c find the probability of a sphere of radius .5 centered at (.5,.5,.5).