Exercise on Probability and Expectation

1. For each of the classes of sets \( \mathcal{G}_i \) below, Show that \( \mathcal{G}_i \) is, or is not, itself a \( \sigma \)-field. If it is not, display the \( \sigma \)-field generated by \( \mathcal{G}_i \). In each case, the underlying complete space \( S = \{1, 2, 3, 4, 5\} \).

\[ \mathcal{G}_1 = \{\{1, 2\}, \{3, 4\}, \{5\}\} \] (1)
\[ \mathcal{G}_2 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\} \] (2)
\[ \mathcal{G}_3 = \{\emptyset, \{1, 3, 5\}, \{2, 4\}, \{1, 2, 3, 4, 5\}\} \] (3)

2. Suppose we define the following function on \( \mathcal{G}_2 \):

\[ P[\{1, 2\}] = .1 \quad P[\{2, 3\}] = .2 \quad P[\{3, 4\}] = .4 \quad P[\{4, 5\}] = .5 \] (4)

(a) Is there a probability defined on the \( \sigma \)-field generated by \( \mathcal{G}_2 \) that matches this function on these sets? Why or why not? If so, are the probabilities of the 5 individual points in \( S \) uniquely determined? If so, what are they?

(b) Answer the question again, this time with

\[ P[\{1, 2\}] = \frac{1}{4} \quad P[\{2, 3\}] = \frac{5}{12} \quad P[\{3, 4\}] = \frac{1}{2} \quad P[\{4, 5\}] = \frac{1}{2} \] (5)

3. (a) Suppose that the random variable \( X \) on \( S \) is defined by

\[ X(1) = 5 \quad X(2) = 4 \quad X(3) = 3 \quad X(4) = 2 \quad X(5) = 1 \] (6)

For each of the \( \mathcal{G}_i \) of problem 1, using a probability \( P \) on \( S \) that you found in problem 2 to be internally consistent, find \( E[X | \mathcal{F}_i] \), where \( \mathcal{F}_i \) is the \( \sigma \)-field generated by \( \mathcal{G}_i \). (This conditional expectation is a random variable on \( S \), of course, so it is a list of 5 numbers, the value of the random variable at each point in \( S \).)
(b) Suppose that the random variable $Z$ on $S$ is defined by

$$Z(1) = 2 \quad Z(2) = 2 \quad Z(3) = 3 \quad Z(4) = 3 \quad Z(5) = 5. \quad (7)$$

Show that $E[X | Z]$ coincides with one of the $E[X | F_i]$ random variables that you computed in part 3a.

4. Consider the functions $F : \mathbb{R}^k \to [0, 1]$ defined by

(a) 

$$F(x, y) = \begin{cases} 
0 & x < 0 \text{ or } y < 0 \\
0.1 + 0.2x \cdot y & x > 0, y > 0, x \cdot y < 4.5 \\
1 & x \cdot y \geq 4.5 
\end{cases}$$

(b) 

$$F(x, y) = \frac{4}{\pi^2} \arctan(e^x) \arctan(e^y)$$

(c) 

$$F(x, y, z) = \begin{cases} 
\min\{x, 1\} \cdot \min\{y, 1\} \cdot \min\{z, 1\} & \text{if } x > 0, y > 0, \text{ and } z > 0 \\
0 & \text{if } x < 0 \text{ or } y < 0 \text{ or } z < 0 
\end{cases}$$

Note that this means that

$$F(x, y, z) = xyz \quad \text{if } x \in (0, 1) \text{ and } y \in (0, 1) \text{ and } z \in (0, 1)$$

In each case, determine whether $F$ is a distribution function. If so, in cases 4a and 4b find the probability of the rectangle with corners (-1,-1) and (3,3), and in case 4c find the probability of a sphere of radius .5 centered at (.5,.5,.5).