Econ. 517, Econometric Theory I — Fall 2001 — Chris Sims

Constructing yield profiles from puts and calls*

1. Mistake

The brief treatment of this in class, which class discussion already showed to be unclear, was actually completely off track. The type of simple combination of underlying asset, put purchase, and call sale described in the lecture delivers not an approximation to the indicator function of the interval $(\bar{p}, \bar{p}+\varepsilon)$ as was claimed, but instead an approximation to a risk free security. A corrected argument is given below.

You are not responsible for being able to reproduce this argument (in this course), and the main point relevant to the rest of the lecture, that arbitrary yield functions $\pi(p)$ can be approximated by combinations of purchases and sales of European puts and calls and the underlying asset, is still correct.

2. Correct argument

A European call option with strike price $\bar{p}$ is an asset with yield, as a function of the price p of the asset itself at the strike date,

$$\gamma(p;\bar{p}) = \begin{cases} 0 & p \leq \bar{p} \\ p - \bar{p} & p > \bar{p} \end{cases} .$$

A European put option with strike price $\bar{p}$ has a yield function

$$\phi(p;\bar{p}) = \begin{cases} \bar{p} - p & p \leq \bar{p} \\ 0 & p > \bar{p} \end{cases} .$$

And of course the underlying asset itself has yield function $\theta(p) \equiv p = \gamma(p;0)$.

Linear combinations of these payout patterns can reproduce exactly any yield function π that consists entirely of linear line segments. To see why, consider first that for any $\bar{p}$, $-\gamma(p;\bar{p}) + \phi(p;\bar{p}) + p \equiv \bar{p}$. In other words, a risk-free yield of $\bar{p}$ can be arranged by buying a put option and the same amount of the underlying stock, while selling the same amount of a call option. Furthermore, given any yield function π, we can change the slope of π by a at all values of $p > c$ by adding to it $a\gamma(\,\cdot\,;c)$. Thus for any π made up of linear line segments, we can match $\pi(0)$ with a risk-free combination of securities, then match $\pi'(0)$ by adding (or selling short) some of the underlying security, then at each node p_i where the slope changes, adding an appropriate (positive or negative) amount of $\gamma(p;p_i)$.

The equations to be solved to find the appropriate amounts of each type of security to match a linear-line-segment π defined by its values $\pi(p_i)$ at a set of points $\{p_i\}_{i=1}^n$ (with $p_0 = 0$ and $\pi(p) \equiv \pi(p_n)$ for $p > p_n$) are given by:

$$\begin{aligned} w &= \pi(0) \\ y &= w + \pi'(0) \\ x_1 &= \pi'(p_1) - \pi'(p_0) \\ &\vdots \\ x_n &= -\pi'(p_{n-1}) , \end{aligned}$$

where w is the amount purchased of a put option with strike price $\pi(0)$, y is the amount purchased of the underlying asset, and x_i, $i = 1, \ldots, n$ is the amount purchased (sold if $x_i < 0$) of a call option with strike price p_i. In these formulas each $\pi'(p_i)$ is the *right* derivative of π at p_i.

It is then easy to see that we could closely approximate the indicator function $1_{(g,h)}(p)$ as a yield function by purchasing (say) 100 call options with strike price $g - .01$, selling 100 call options with strike price g, selling 100 call options with strike price h, and buying 100 call options with strike price $h + .01$. We can make the approximation arbitrarily good by replacing 100 and .01 in this prescription with ψ and $1/\psi$, and taking ψ very large.

EXERCISES TO CHECK YOUR UNDERSTANDING:

You do not need to hand this in, but if you can't figure out how to go about answering these questions, ask.

(a) The stock price is currently \$100 per share. A put option to sell it a week from now at \$50 per share is priced at \$3.75. A call option to buy it a week from now at \$50 is priced at \$54. What is the risk free interest rate?

(b) In addition to the prices given in the previous question, suppose we know that we have the following list of call option prices for the same date:

strike price	market price
49	55.00
51	53.05
59	45.00
60	44.50
61	44.05

Use this information to put upper and lower bounds on the price of a security that pays \$10 if the stock price at the strike date is between \$50 and \$60, and pays zero otherwise (i.e. a security with payout $\pi = \mathbf{1}_{(50,60)}$). What do these imply as bounds on $P[\$50 < p < \$60]$, where p is the stock price at the strike date and P is the market probability as discussed in class?