Probability examples; Distribution Functions

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Generating probabilities

- Discrete probability: finitely many points in the space:

$$\mathcal{S} = \left\{ x_j,\ j = 1, \ldots, n \right\}$$

$$p : \mathbb{Z}^+ \to [0,1], \quad \sum_j p_j = 1$$

$$P[A] = \sum_{\left\{ j : x_j \in A \right\}} p_j$$

Examples: $\quad p_j = .5,\ j = 1, 2$

$$p_j = 2^{-j},\ j = 1, \ldots, \infty$$

- Integrating densities, on $\mathbb{R}$:

$$\mathcal{S} = \mathbb{R}; \quad p : \mathbb{R} \to [0, \infty)$$

$$\int p(x)\,dx = 1$$

$$P[(a,b)] = \int_a^b p(x)\,dx$$

Examples: $\quad p(x) = \frac{1}{2} e^{-|x|}$

$$p(x) = \mathbf{1}_{(0,1)}(x)$$

- On $\mathbb{R}^k$:

$$\mathcal{S} = \mathbb{R}^k; \quad p : \mathbb{R}^k \to [0, \infty)$$

$$\int p(\vec{x})\, d\vec{x} = 1$$

$$(\vec{a}, \vec{b}) = \left\{ \vec{x} \in \mathbb{R}^k : a_i < x_i < b_i,\ i = 1, \ldots, k \right\}$$

$$\Rightarrow P[(\vec{a}, \vec{b})] = \int_{(\vec{a}, \vec{b})} p(x)\, dx$$

$$= \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} p(x_1, \ldots, x_k) dx_1 dx_2 \cdots dx_k$$

Examples: $\qquad p(x, y) = .25 e^{-|x| - |y|}$

$$p(x, y) = 1.5 \max\left\{1 - |x| - |y|, 0\right\}$$

- Distribution functions on $\mathbb{R}$:

$$\mathcal{S} = \mathbb{R}, \quad F : \mathbb{R} \to [0, 1]$$

$$a > b \Rightarrow F(a) \geq F(b); \quad \lim_{x_n \downarrow x} F(x_n) = F(x)$$

$$P\Big((-\infty, a]\Big) = F(a)$$

Examples: $F(x) = \begin{cases} 0 & x < 0 \\ x & 0 < x < 1 \\ 1 & x > 0 \end{cases}$

$$F(x) = \frac{e^x}{1 + e^x}$$

- Distribution functions on $\mathbb{R}^k$:

$$\mathcal{S} = \mathbb{R}^k$$

$$F : \mathbb{R}^k \to [0,1], \quad \lim_{\vec{x}_n \downarrow \vec{x}} F(\vec{x}_n) = F(\vec{x})$$

$$P[\left\{x \in \mathbb{R}^k | x_i \leq a_i,\ i = 1, \ldots, k\right\}] = F(a)$$

$$\frac{\partial^k F}{\partial x_1 \ldots \partial x_k} \geq 0 \text{ If it exists.}$$

More generally, letting

$$\Delta_i^{\varepsilon} f(x) = f(x_1, \ldots, x_i + \varepsilon, x_{i+1}, \ldots, x_n) - f(\vec{x}),$$

$$(\forall \vec{\varepsilon} \geq 0) \Delta_1^{\varepsilon 1} \Delta_2^{\varepsilon 2}, \ldots \Delta_n^{\varepsilon n} F(x) \geq 0$$

Examples:

$$F(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ .1 + .2x \cdot y & x > 0,\ y > 0,\ x \cdot y < 4.5 \\ 1 & x \cdot y \geq 4.5 \end{cases}$$

$$F(x,y) = \frac{4}{\pi^2} \arctan(e^x) \arctan(e^y)$$