FINAL EXAM

The exam is due at 9AM Tuesday, January 24. A well-prepared student should be able to do it in under four hours. It should be turned in electronically to umueller@princeton.edu and sims@princeton.edu. If you need to arrange turning in a paper exam answer, email sims.princeton.edu before 4PM Monday to arrange this. Unlike on exercises, on the exam you are not to collaborate with others. Do not discuss the exam with anyone before the due time.

1. (30 points) Consider the bivariate observation

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu + \Delta \\ \mu \end{pmatrix}, I_2 \right)$$

where $\mu \in \mathbb{R}$ and $\Delta \geq 0$. Further define the group of transformations

$$X \to X + \begin{pmatrix} c \\ c \end{pmatrix}$$

for $c \in \mathbb{R}$.

(a) (10 pts) Find a maximal invariant to this group of transformations, and show that it satisfies the definition of a maximal invariant.

(b) (10 pts) Suppose we are interested in $H_0 : \Delta = 0$ against $H_1 : \Delta > 0$. Derive the best invariant test. Is this test biased?

(c) (10 pts) Suppose we are interested in $H_0 : 0 \leq \Delta \leq 1$ against $H_1 : \Delta > 1$. Derive the best invariant test. Is this test biased?

2. (30 points) Suppose

$$y_t = \varepsilon_t + \frac{g}{T} \sum_{s=1}^{t} \eta_s$$

where $(\varepsilon_t, \eta_t) \sim iid(0, \sigma^2 I_2)$. For some fixed integer q, let

$$X_{T,j} = T^{-1/2} \sum_{t=\lfloor (j-1)T/q \rfloor + 1}^{\lfloor jT/q \rfloor} y_t, \quad j = 1, \ldots, q$$

that is $X_{T,j}$ is the scaled average of the jth block of $[T/q]$ values of y_t.

(a) (12 pts) Show that $X_T = (X_{T,1}, \ldots, X_{T,q})'$ converges in distribution to $X \sim \mathcal{N}(0, \Sigma)$, where Σ is a function of g and σ. (You do not need to find an expression for Σ, but please provide an argument for why X is multivariate normal.)

(b) (8 pts) What is Σ if g is zero?

(c) (7 pts) Let $\bar{X}_T = q^{-1} \sum_{j=1}^{q} X_{T,j}$ and $s_T^2 = (q-1)^{-1} \sum_{j=1}^{q} (X_{T,j} - \bar{X}_T)^2$. What is the limiting distribution of the statistic $\sqrt{q}\bar{X}_T / s_T$ if $g = 0$?

Date: January 23, 2017.

3. (30 points) Suppose we observe a random sample of size n of i.i.d. Bernoulli(p) random variables X_i.

(a) (15 pts) Show that the model with $p = p_n = p_0 + g/\sqrt{n}$ for fixed g is contiguous to the model with $p = p_0$, for $p_0 \in (0, 1)$.

(b) (15 pts) Show that the model with $p = p_n = g/n$ for fixed g is contiguous to the model with $p = p_n = \lambda/n$, for fixed $\lambda > 0$. Hint: Recall that under $p = \lambda/n$, $S_n \Rightarrow Y$, where Y is Poisson with parameter λ, and the moment generating function of Y is $E[e^{tY}] = \exp(\lambda e^t - \lambda)$.

4. (90 points) Here is a sequence of 20 observations, ordered left to right, first row, then second row:

1.65	−0.34	−1.58	−0.41	0.18	−0.84	1.17	−0.87	−0.87	1.06
0.41	−1.68	1.40	−1.44	−0.36	1.16	0.81	−0.99	0.31	0.76

They were generated from a model of the form

$$y_t = by_{t-1} + \varepsilon_t - a\varepsilon_{t-1}\,, \quad |a| < 1\,, \quad |b| < 1\,,$$

where the ε_t values are i.i.d. $N(0, 1)$.

(a) Write a program to evaluate the log likelihood for this model as a function of a and b. Use the full unconditional distribution of the y vector; do not condition on initial conditions. This requires finding the unconditional variance. One approach is to convert the model to a first order bivariate AR and apply a doubling algorithm, but there are other approaches. Your program should be turned in with your exam.

(b) Plot the level curves of the log likelihood and the likelihood itself in (a, b) space.

(c) This model has an identification problem. Explain what the problem is and how it shows up in your plots.

(d) Does the identification problem imply that attempts to draw from the likelihood as a posterior distribution using an MCMC algorithm would fail to converge? Why or why not?

(e) The true values of the parameters that generated this y sequence were $a = .9$, $b = .8$. With this sample, would the Schwarz criterion reject a model with a and b restricted to these values in favor of the model with a and b unknown? What about a model with $a = b = 0$?