Frequency Domain Models

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Complex Gaussian variates

A **complex Gaussian** (or complex normal) random variable is a random variable of the form $X + iY$, where X and Y are a pair of random variables jointly distributed as $N(\mu, \sigma^2 I)$ for some σ^2. One might think that this name should apply also to random variables in which the real and imaginary parts have different variances, or are correlated, but it turns out to be convenient to reserve the term for this more restricted definition. If X is a jointly complex normal vector, we define $\mathrm{Cov}(X) = E[(X - E(X))(X - E[X])']$, where $'$ denotes transposition and complex conjugation. A covariance matrix Σ for a complex-valued random variable is therefore always Hermitian (that is, $\Sigma = \Sigma'$).

Frequency-domain representation of a stationary process

We have already observed that the finite Fourier transform represents a vector as a linear combination of sine and cosine functions. If $\bar{X}_T = \{X_1, \ldots, X_T\}$ are a sequence of values drawn from a stochastic process, then $\tilde{\bar{X}}_T = FX_T$ can be thought of as random weights on the periodic functions $T^{-1/2}\exp(2\pi ijk/T)$, $j = 0, \ldots, T-1$ that make up the columns of F'. That is, we can write $\bar{X}_T = F'\tilde{\bar{X}}_T$.

Spectral Density

- The **spectral density** S_X of a stationary process X with autocovariance function R_X is defined by $S_x = \tilde{R}_X$. That is, the spectral density is the Fourier transform of the autocovariance function.

- If $Y_t = a(L)X_t$, then we know that $R_Y(L) = a(L)R_X(L)a'(L)$. But then by the fact that FT tranforms convolution or multiplication of polynomials in the lag operator into ordinary multiplication in the frequency domain, $S_Y = \tilde{a} \cdot S_X \cdot \tilde{a}'$, where as usual in the frequency domain, $'$ both transposes and takes complex conjugates of elements.

The Fourier transform of a stochastic process

- We have already seen that we can think of the expression $X = \int a(s)dW(s)$ as defined by the properties that X is normal and $\mathrm{Cov}(\int a(s)dW(s), \int b(s)dW(s)) = \int a(s)b'(s)ds$.

- We can make a similar indirect characterization of the Fourier transform $\tilde{X}(\omega) = dZ_X(\omega)$ of a stationary process X by saying that for any square-integrable functions $\tilde{a}(\omega)$, $\tilde{b}(\omega)$, $Y = \int \tilde{a}(\omega)dZ_X(\omega)$ and $U = \int \tilde{b}(\omega)dZ_X(\omega)$ are jointly normal with covariance matrix $\int \tilde{a}(\omega)S_X(\omega)\tilde{b}'(\omega)d\omega$.

- Then we can take $\sum a(s)X(s) = \int \tilde{a}(\omega)dZ_X(\omega)$, where a is the inverse FT of $\tilde{a}$. In continuous time, this relation is instead $\int a(x)X(s)ds = \int \tilde{a}(\omega)dZ_X(\omega)$. The frequency domain limits of integration are $(-\pi, \pi)$ for discrete time and $(-\infty, \infty)$ for continuous time.

- As with a Wiener process, where dW is "white noise" that is undefined at single points of time, dZ_X also is undefined at single points ω.

- $Z_X(\omega) = \int_{-\pi}^{\omega} dZ_X(\nu)$, like $W(t)$, has independent increments, since if a and b are indicator functions (or vectors thereof) for non-overlapping intervals, $\int a S_X b = 0$.

- The differences between Z_X and W are that Z_X is complex Gaussian, not Gaussian, and that the variance of $Z_X(\omega + \delta) - Z_X(\omega)$ varies with ω, indeed is approximately $\delta S_X(\omega)$ for small δ at continuity points of S_X.

Relation of FFT to DFT

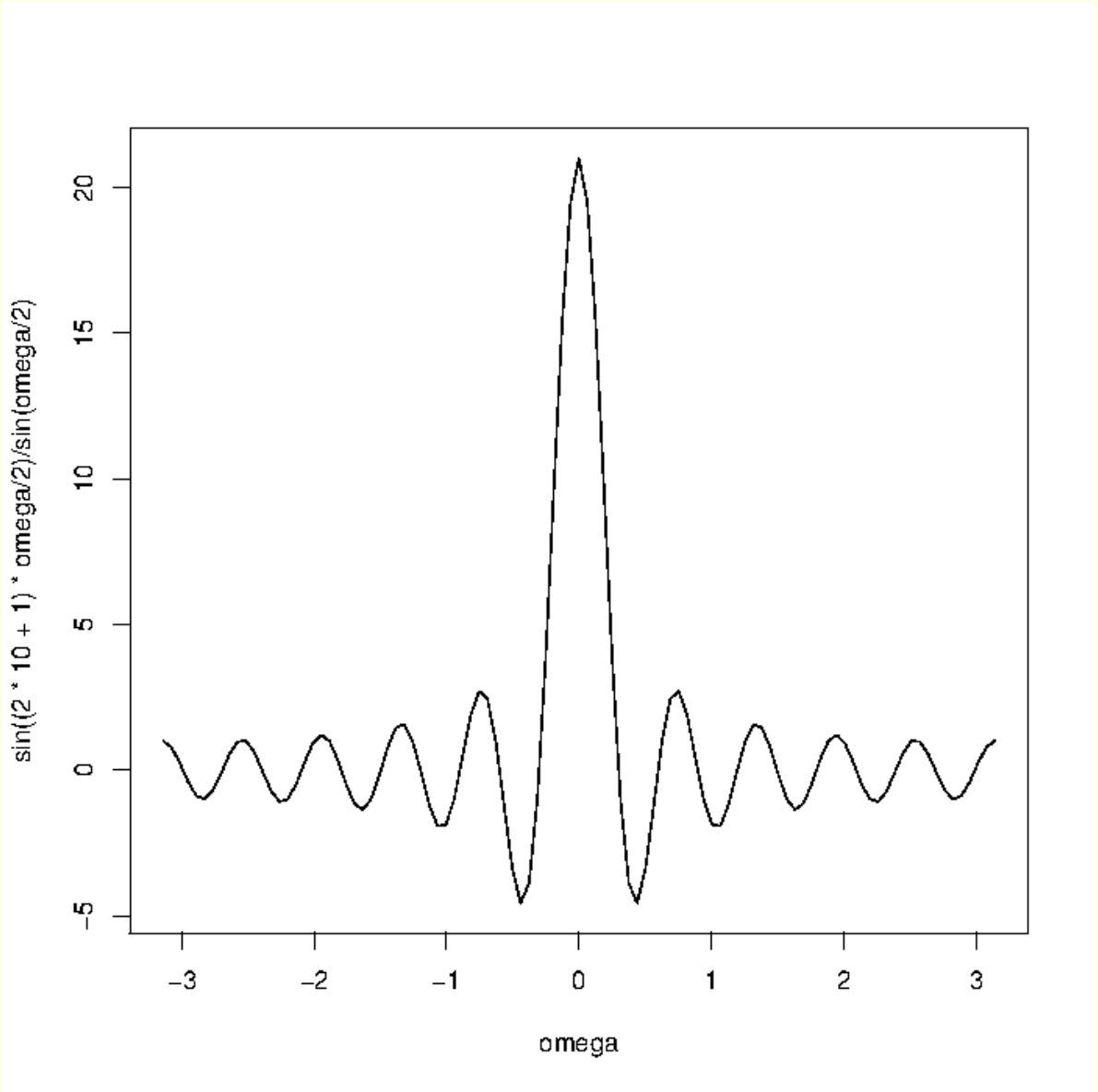
- FFT of $\{X_1, \ldots, X_T\}$ can be regarded as DFT of $\{X_t,\ t = -\infty, \ldots, \infty\} \times \mathcal{I}_{[0,T]}(T)$.

- To avoid keeping track of time shifts, let's consider FFT of $\{X_t,\ t = -T, \ldots, T\}$. If $a = \mathcal{I}_{[-T,T]}$, then the DFT of $a \times \{X_{-\infty}, \ldots, X_\infty\}$ is

$$\tilde{a} * \tilde{X}(\omega) = \int_{-\pi}^{\pi} \tilde{a}(\omega - \nu) dZ_X(\nu)\,.$$

- $$\tilde{a}(\omega) = \frac{e^{i\omega T} - e^{-i\omega(T+1}}{1 - e^{-i\omega}} = \frac{\sin((2T+1)\omega/2}{\sin(\omega/2)}$$

- So at $\omega_j = 2\pi j/(2T+1)$, $\tilde{a}$ is zero, except at $j = 0$, where (by l'Hôpital's rule) it is $2T+1$. See plot.


sin((2 * 10 + 1) * omega/2)/sin(omega/2)
20
15
10
5
0
−5
−3
−2
−1
0
1
2
3
omega

Properties of the FFT of X

- The connection to the DFT and the shape of $\tilde{a}$ plotted above suggest that the FFT of $X_1, \ldots, X_T$ should show approximate independence, at least over gaps of at least length $2\pi/T$ and for large T.

- If X is circular (so that the covariance of X_t with $X_{\bmod (t-s,T)}$ does not depend on t), then $\tilde{X}$ is *exactly* mutually independent.

- If X is merely stationary, then this result holds approximately, in large samples.

- So: a nonparametric estimate of the spectral density $S_X(\omega)$ is obtained by averaging $\left|\tilde{X}\right|^2$ over a number of adjacent values of $2\pi j/T$ near ω. These can be weighted in various ways. $\left|\tilde{X}\right|^2$ is called the **periodogram** of X.

The Daniell window

- If we estimate the spectral density at ω by simply averaging the k periodogram ordinates at harmonic frequencies $2\pi j/T$ nearest to ω, we are using what is called a "Daniell window".

- The Daniell window is equivalent to multiplying the sample autocovariance function $\hat{R}_X(t) = T^{-1}\sum_1^{T-t} X_s X'_{t+s}$ by our familiar

$$\frac{T^2 \sin(2\pi k t/T)}{t(2\pi k)^2}$$

  This is $T/(2\pi k)$ at $t = 0$ and vanishes at $t = T/(2k)$. Thus it can be thought of as damping down $\hat{R}_X$ at large values of t. All the other "windows" used in spectral density estimation do this also.