WIENER PROCESS; FOURIER ANALYSIS

1. Convolution

- Continuous
$$f * g(t) = \int f(s)g(t-s)\,ds = \int g(s)f(t-s)\,ds$$
- Discrete
$$f * g(k) = \sum_j f(j)g(k-j) = \sum_j g(j)f(k-j)$$
- Circular
$$f * g(k) = \sum_{j=0}^{n-1} f(j)g\big(\mathrm{mod}(k-j,n)\big) = \sum_{j=0}^{n-1} g(j)f\big(\mathrm{mod}(k-j,n)\big)$$

  Here $\mathrm{mod}(a,b)$ is a nonn-negative number less than b such that $a = kb + \mathrm{mod}(a,b)$ for some integer k. Thus $\mathrm{mod}(-3,8) = 5$, $\mathrm{mod}(17,3) = 2$, and $\mathrm{mod}(8.2,3.6) = 1.0$, for example.

2. Fourier transform

$$e^{a+bi} = e^a\big(\cos(b) + i\sin(b)\big)$$

Continuous: $L_2 \leftrightarrow L_2$
$$\tilde{f}(\omega) = \int f(s)e^{-i\omega s}\,ds$$

Discrete: $L_2(0,2\pi) \leftrightarrow \ell_2$
$$\tilde{f}(\omega) = \sum f(s)e^{-i\omega s}$$

Circular, or finite: $\mathbb{R}^n \leftrightarrow \mathbb{R}^n$
$$\tilde{f}\left(\frac{2\pi k}{n}\right) = n^{-\frac{1}{2}} \sum_{j=0}^{n-1} f(j)\exp\left(-\frac{2\pi ijk}{n}\right)$$

With F the $n \times n$ matrix with j'th row, k'th column entry
$$n^{-\frac{1}{2}}\exp(-2\pi i(j-1)(k-1)/n)\,,$$
this can be written as $\tilde{f} = Ff$, where f is now interpreted as the column vector of f's distinct values. For complex matrices we will interpret $'$, the transpose operator, as taking complex conjugates of matrix entries as well

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as transposing. Then $F'F = I$, and F' is the inverse finite Fourier transform operator.

3. Why the Fourier transform is useful

- $\widetilde{f * g} = \tilde{f} \cdot \tilde{g}$
- If a vector x of length n is exactly periodic with integer period n/k, k a positive integer, then $\tilde{x}$ will be zero except at the points $2\pi jk/n$ for integer j. In other words, the finite FT allows us to express any function that is periodic with period p as a linear combination of sine and cosine functions with periods p/j, for integer j.

4. Examples of FT's

$$
\begin{array}{rl}
e^{-rs}, s > 0 & \dfrac{1}{i\omega + r} \\
e^{-r|s|} & \dfrac{1}{i\omega + r} + \dfrac{1}{-i\omega + r} = \dfrac{2r}{\omega^2 + r^2} \\
e^{-\frac{1}{2}s^2} & \sqrt{2\pi} e^{-\frac{\omega^2}{2}} \\
\text{tent} = 1 - |s|\, on(-1, 1) & \left| \dfrac{1 - e^{-i\omega}}{\omega} \right|^2 = \dfrac{2 - 2\cos(\omega)}{\omega^2} \\
= \text{box} * \text{box}' &
\end{array}
$$

5. FT's of derivatives, lag operators, combs — stochastic processes?!

6. Discontinuity ↔ decay rates.

7. Wiener Process

- Fully characterized as a Gaussian process W_t with stationary, independent increments and with $\text{Var}(W_{t+1} - W_t) = 1$ and (as a convention) $W(0) = 0$.
- If a is a non-stochastic function of time with $\int a(s)^2\, ds < \infty$, $\int a(s) dW(s)$ is well defined as a Gaussian random variable with mean 0, variance $\int a^2$. If we form two random variables this way, using functions a and b and the same Wiener process for both, the covariance of the two random variables is $\int a(s)b(s)\, ds$.
- The Wiener process can be chosen to have continuous time paths a.s., but must be a.s. nowhere differentiable w.r.t. time.

8. Continuous time Gaussian MA processes

- $X_t = \int a(t - s) dW(s)\, ds$ is a continuous time MA process.
- Sometimes we write $\varepsilon(t)dt$ in place of $dW(t)$. Then $\varepsilon(t)$ is "continuous time white noise".
- Same problem as in discrete case of multiple MA rep's of same process.

- Definition of fundamental is the same: σ-field generated by X_s, $s \leq t$ must be same as that generated by W_s, $s \leq t$.
- For fundamental rep, $\int_0^\varepsilon a(s)dW(t-s)$ is the ε-step ahead forecast error in forecasting X_{t+s} based on $\{X_v, v \leq t\}$.

9. Other CT↔DT analogies

$$R_x(s) = a * a'(s) \tag{1}$$

$$x \text{ linearly regular } \Leftrightarrow \int_{-\infty}^{\infty} \frac{\log(S_x(\omega))}{1+\omega^2} > -\infty \tag{2}$$

$$x = a * dW \quad \text{ fundamental } \Leftrightarrow \tilde{a}(\omega) \neq 0\,, \quad \forall\, \text{Real}(\omega) < 0 \tag{3}$$

$$a(s) = e^{-\rho s}\,, \quad x = a * dW \quad \Rightarrow \quad R_x(t) = e^{-\rho|t|} \tag{4}$$

$$(D+\rho)x = dW \quad \text{ or } \dot{x} = -\rho x + \varepsilon\,. \tag{5}$$

This is an "Ohrnstein-Uhlenbeck" process, the continuous time version of a first-order AR. Sampled at unit time intervals, it is an AR with paramter $e^{-\rho}$.

We can interpret ρ in (4) and (5) as a matrix. In (5) it is pretty clear what this means. But in (4) it is not so clear. What is e^A when A is a matrix? In R, the expression `exp(A)` is just a matrix whose typical element is $e^{a_{ij}}$, while in Matlab it is the **matrix exponential** of A. The matrix exponential can be defined by a series expansion:

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

When A has distinct eigenvalues, so that $A = PDP^{-1}$ with D diagonal, we can also write

$$e^A = P\,\text{diag}(e^{d_1}, \ldots, e^{d_n})P^{-1}\,,$$

where $\text{diag}(v)$ is the diagonal matrix with diagonal entries given by v and we have assumed $A = P\,\text{diag}(d)P^{-1}$. Computing a matix exponential in R requires an externally supplied function. There is a matlab library called `expokit`, which I have converted to R, which does this.