

FREQUENCY DOMAIN EXERCISE

- (1) Consider a process with spectral density $S_x(\omega)$ that takes the value 1 at ω equal to $0, \frac{\pi}{2}, \frac{3\pi}{2}, \pi, \text{etc.}$, the value 0 at $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \text{etc.}$, and at other values of ω consists of linear line segments connecting these points. I.e., it is a repeated sequence of “tents” of height one and base $\frac{\pi}{2}$, centered on the points $\frac{2\pi k}{4}$, all integer k . Display the fundamental moving average operator. [Hint: Find R_x . Recall that the FT of a sequence of equal-height spikes at $\delta_j, j = -\infty, \dots, \infty$ is a sequence of equal-height spikes in the frequency domain at $\frac{2\pi j}{\delta}, j = -\infty, \dots, \infty$. Of course this S_x is not simply a sequence of spikes...]

If we think of the spectral density as the convolution of spikes of height 2π at the seasonal frequencies $\frac{\pi k}{2}$ convoluted with a “tent” of width $\frac{\pi}{2}$ and height 1, its inverse FT (i.e. the autocovariance function) is the product of the IFT's of these two components. The IFT of the spikes is itself a set of spikes, at intervals $\frac{2\pi}{(\pi/2)} = 4$. The height of the spikes is $\frac{1}{2\pi}$ times the height of the spikes times the number of spikes in each $(2k\pi, 2k\pi + 2\pi)$ interval (i.e. 4 in this case), so the height of the time domain spikes is 4. The tent with base $\frac{\pi}{2}$ and height 1 is a “box” of height $\sqrt{8}$ on $0, \frac{\pi}{4}$ convoluted with itself with reversed time index, so its IFT is the squared absolute value of the IFT of the box:

$$\frac{1}{4\pi^2} \cdot 8 \frac{2 - 2 \cos(\pi t/4)}{t^2}.$$

Evaluated at $t = 0$ using l'Hopital's rule, this is $\frac{1}{8}$. So at $t = 0$ we get $\frac{1}{2}$ as $R_x(0)$. This is good, since we can easily check that the IFT of the spectral density, being $\frac{1}{2\pi}$ times its integral over $(-\pi, \pi)$ is in fact $\frac{1}{2}$. Notice that in calculating the scale factors here I'm using the fact that in the frequency domain for FT's of discrete-time processes on the whole of \mathbb{Z} , convolution has to be defined as

$$f * g(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(v)g(\omega - v)dv.$$

Without the $\frac{1}{2\pi}$ factor the usual identity saying that the IFT of a convolution is the product of the IFT's would not work. This is not something I pointed out in class, which may be why there was a lot of diversity in answers about the scale factor.

A quick and dirty way to make this work would be to recognize a) that since the integral of the spectral density is π , the IFT at 0 has to be $\frac{1}{2}$. Since the R_x function is going to be the product of spikes at intervals of 4 with some scaling of $(2 - 2 \cos(\pi t/4))/t^2$, one can just use our knowledge of what the product has to be at $t = 0$ to scale the whole thing, without keeping track of scale factors in all the intermediate calculations.

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In any case, the final result for R_x is

$$R_x(t) = \frac{2 - 2 \cos(\pi t/4)}{t^2} \cdot \frac{8}{\pi^2}$$

$$= \{.5, .2026, 0, .0225, 0, .0081, 0, .0041, 0, .0025, 0, .0017\} \text{ at } t = \{0, 4, 8, \dots, 48\} .$$

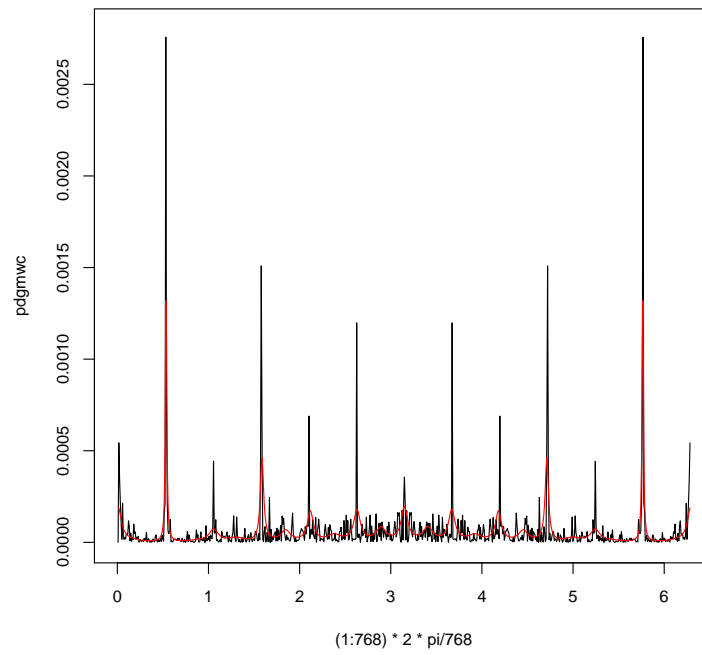
The first few coefficients in the MAR, found by

```
rx4 <- c(.5, (8 / pi^2) *(2-2 * cos(pi * t / 4)) / t^2 )
rx <- vector("numeric", 257)
rx[seq(1, 257, by=4)] <- rx4
M <- toeplitz(rx)
cM <- chol(M)
cM[ , 257]
```

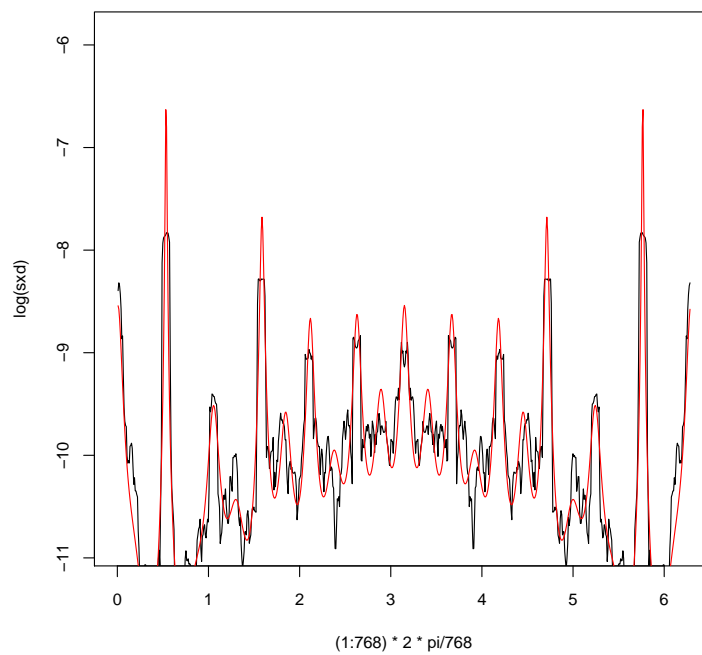
- are .608, 0,0,0, .356, 0,0,0, -.306, 0,0,0, .047, 0,0,0, -.013, 0,0,0, .019, 0,0,0, -.008.
- (2) Data on the seasonally unadjusted average hourly wage in the construction industry is available on the course web site. In this exercise you first preprocess the data, using first differences of the log of the raw data, so we can focus on seasonality.
- Form the periodogram of the series and plot it, and its log. Pad the series to length $3 \cdot 2^8$ so as to make the fft efficient.
 - Form a Daniell-window estimate of the spectral density, using a window width of 11 periodogram ordinates. Plot the result and its log. [The `filter()` function in R or Matlab will help with this.]
 - Fit a 24-th order AR model (including a constant term) to the data and plot the log of the spectral density implied by the fitted coefficients. [The `rfvar3` program, in R or Matlab, makes this easy, but it is also a straightforward linear regression. Priors will not matter much here, but you should probably set the `lambda`, `mu` parameters in `rfvar3` to 0 if you use that, as the built-in prior in that program ignores and therefore damps down seasonality.]
 - Extract a deterministic seasonal from the data by taking averages of the data for each month over the whole sample. Subtract this deterministic seasonal from the data and repeat parts 2a and 2b for this deterministically adjusted series. Comment on whether the deterministic adjustment seems adequate.

I think everyone had the right general idea here. As in the first problem, there are issues as to how to scale things to get comparable AR and frequency-domain estimates. I estimated the AR model by OLS, with a constant, and obtained a pretty good match to the periodogram, as shown in the two graphs below. The first is levels, the second is logs.

Periodogram and AR spectral density, levels



log spectral density, Daniell and AR (red)



Note that I removed the sample mean from the first differenced log wage data before forming the periodogram. I also divided the squared FFT of the wage series by the length of the series. For the FFT of the AR operator, I did not divide by sample size. The spectral density should be the FT of the autocovariance function. As we went through in class, the squared periodogram can be thought of as the FT of sample size

times the autocovariance function. The 24-lag AR regression provides a surprisingly good match to the Daniell window estimate.