

## MULTIVARIATE ARMA, KALMAN FILTER

### 1. THE FINITE MA CLASS OF MODELS

$$y_t = \sum_{s=0}^k a_s \varepsilon_{t-s} = a(L) \varepsilon_t .$$

$y$  may be  $m \times 1$ , in which case  $a_s$  is  $m \times m$ .  $\varepsilon \sim N(0, \Sigma)$ , i.i.d. Or sometimes just mean 0, variance  $\Sigma$ , not serially correlated.

Properties:

- Dense in the space of LR stationary processes.
- Closed under taking linear combinations.
- Closed under taking subvectors.
- To keep uniqueness, must restrict parameter space to fundamental MA's. This restriction (on roots) is quite nonlinear. But the fundamental MA's form a closed set with open interior, since roots are continuous functions of parameters.

### 2. IS THE SET OF FUNDAMENTAL MA OPERATORS CONVEX?

This is an important point, since if it is convex, then iterative methods for maximizing likelihood subject to the constraint are likely to work well, because concave functions over a convex set have a unique maximum, while with a non-convex constraint set there can be multiple boundary maxima, even for concave functions.

For second-order operators, the set is convex. There is a famous diagram showing the set of values of  $\rho_1$  and  $\rho_2$  for which the roots of  $1 + \rho_1 z + \rho_2 z^2$  all lie on or outside the unit circle, shown in Figure 1. This is obviously a convex region.

However, beyond two dimensions the region is no longer convex. For example, consider  $P(L) = 1 - 3L + 3L^2 - L^3$  and  $Q(L) = 1 + L$ . The first has three roots, all 1. The second has one root of -1. An equal weighted linear combination of the two,  $1 - L + 1.5L^2 - .5L^3$  has a pair of complex roots with absolute value .89, and one real root of 2.5. So the region is not convex for third order polynomials.

### 3. THE FINITE AR CLASS OF MODELS

$$y_t = \sum_{s=1}^k b_s y_{t-s} + \varepsilon_t, \quad \text{or } b(L)y_t = \varepsilon_t .$$

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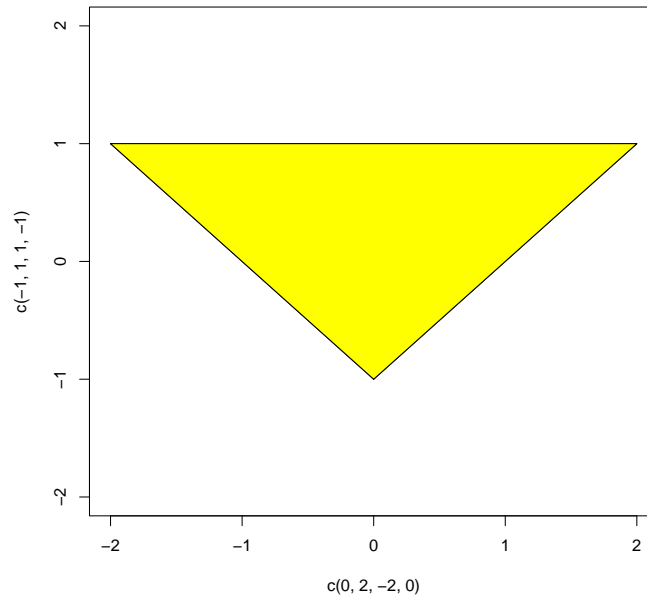


FIGURE 1.  $\rho_1, \rho_2$  values yielding invertible  $1 + \rho_1 L + \rho_2 L^2$

$\varepsilon \sim N(0, \Sigma)$ , or sometimes just mean 0, variance  $\Sigma$ , not correlated with past  $y$ 's, and therefore not serially correlated.

Properties:

- Dense in the space of LR stationary processes, plus includes some types of non-stationary processes
- *Not* closed under taking linear combinations
- *Not* closed under taking subvectors.
- No uniqueness problem. Every set of real numbers used to populate  $b_s$ ,  $s = 1, \dots, k$  results in a distinct model. Restrictions like that to obtain fundamental MA's if we want to consider only stationary models. But this restriction is not needed to prevent redundancy.

#### 4. FINITE-ORDER ARMA MODELS

$$B(L)y_t = A(L)\varepsilon_t,$$

where  $\varepsilon_t \perp \{y_s, s < t\}$  (and  $\varepsilon_t$  is therefore the innovation in  $y$  at  $t$ ) and  $B$  and  $A$  are finite-order polynomials in  $L$ , perhaps with matrix-valued coefficients.

Properties:

- Contains MA and AR models, so is also dense in the LR class of models.
- Closed under taking linear combinations.

- Like the finite-order AR class, contains non-stationary as well as stationary models.
- Has the same problems as the MA class with possible redundancy in the  $A(L)$  parameter space.
- Has the same problem as the AR class with the restrictions on  $B(L)$  needed if we want to restrict to stationary models.
- Has its own special, severe problem of non-uniqueness, because of possible cancellation between AR and MA roots.

*Practice exercise:* Suppose  $y_t = .7y_{t-1} + \varepsilon_t$ ,  $z_t = .9z_{t-1} + v_t$ , where  $\varepsilon_t$  and  $v_t$  are i.i.d.  $N(0, I)$  and are uncorrelated with all  $y_s, z_s$  for  $s < t$ . (This makes them the innovations in the joint  $y, z$  process, of course.) Find the fundamental univariate MA representation for  $x_t = y_t + z_t$ . [Hint: This will take the form  $(P(L)/Q(L))\eta_t$ , where  $\eta_t$  is the univariate innovation in  $x_t$ . Form the acf of  $x$ , as a ratio of polynomials in the lag operator, and then factor to get expressions in positive powers of  $L$  with no roots inside the unit circle.]

## 5. THE KALMAN FILTER

Model in the form

Plant equation:  $s_t = As_{t-1} + \varepsilon_t$

Measurement equation:  $y_t = Hs_t + v_t$ .

$\text{Var}(\varepsilon_t) = \Omega$ ,  $\text{Var}(v_t) = \Xi$ .  $\varepsilon_t \perp v_t$  and  $(\varepsilon_t, v_t)$  i.i.d., independent of past  $y, s$ .

KF: A rule for starting with a prior  $s_t \sim N(\mu_t, \Sigma_t)$ , using it, plus observation of  $y_{t+1}$ , to update to a new distribution  $s_{t+1} \sim N(\mu_{t+1}, \Sigma_{t+1})$ .

Note that the tradition of having a separate error term  $v_t$  in the observation equation is unnecessary. We can relabel any such shocks as elements of  $s$ , after which there are no errors in the observation equation. That is, we can define

$$u_t = \begin{bmatrix} s_t \\ r_t \end{bmatrix}$$

and then rewrite the model as

$$\begin{aligned} u_t &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} u_{t-1} + \begin{bmatrix} \varepsilon_t \\ v_t \end{bmatrix} \\ y_t &= [H \quad I] u_t. \end{aligned}$$

$$\left\{ \begin{bmatrix} s_{t+1} \\ y_{t+1} \end{bmatrix} \mid \mathcal{I}_t \right\} \sim N \left( \begin{bmatrix} A\mu_t \\ HA\mu_t \end{bmatrix}, \begin{bmatrix} A\Sigma_t A' + \Omega & A\Sigma_t A' H' + \Omega H' \\ HA\Sigma_t A' + H\Omega & HA\Sigma_t A' H' + H\Omega H' + \Xi \end{bmatrix} \right).$$

Applying the formula for the conditional distribution of one Gaussian random variable given another, we get

$$\begin{aligned} \{s_{t+1} \mid \mathcal{I}_{t+1}\} &\sim N(\mu_{t+1}, \Sigma_{t+1}) \\ \mu_{t+1} &= A\mu_t + (A\Sigma_t A' H' + \Omega H')(HA\Sigma_t A' H' + H\Omega H' + \Xi)^{-1}(y_{t+1} - HA\mu_t) \\ \Sigma_{t+1} &= A\Sigma_t A' + \Omega \\ &\quad - (A\Sigma_t A' H' + \Omega H')(HA\Sigma_t A' H' + H\Omega H' + \Xi)^{-1}(HA\Sigma_t A' + H\Omega) \end{aligned}$$

Note that, though this looks like messy algebra, if  $y_t$  is a scalar, there is no matrix inversion involved. It is worthwhile consulting numerical analysis literature or using an optimized program if you want to do this with large matrices or many times.

## 6. DATA DENSITY

At each date, the Kalman filter involves forming a normal distribution for  $y_{t+1} \mid \mathcal{I}_t$ . Calling the pdf of this distribution  $p(y_{t+1} \mid \mathcal{I}_t)$ , the pdf of the entire observed sample of  $y$ 's is then

$$\prod_{t=1}^T p(y_t \mid \mathcal{I}_{t-1})$$

This formula applies because we assume the information available at  $t$  consists of the time zero information  $\mathcal{I}_0$  plus the sequence of  $y_s$  values for  $s \leq t$ . The Kalman filter only tells us how to derive  $p(\cdot \mid \mathcal{I}_{t+1})$  from  $p(\cdot \mid \mathcal{I}_t)$  and  $y_t$ . The initial distribution  $p(\cdot \mid \mathcal{I}_0)$  is determined by an initial Gaussian prior on the initial state  $s_0$ .

## 7. LIKELIHOOD

The log posterior density (often imprecisely called the log likelihood, despite the fact that it involves a prior density) is then just the sum of the  $\log(p(y_t \mid \mathcal{I}_t))$  terms. A single one of those terms is

$$-\frac{1}{2}(y_t - \hat{y}_t)' \Phi_t^{-1} (y_t - \hat{y}_t) - \frac{1}{2} \log |\Phi_t| ,$$

where  $\hat{y}_t = HA\mu_{t-1}$  and  $\Phi_t = HA\Sigma_{t-1}A'H' + H\Omega H' + \Xi$  are the mean and variance matrix of the one-step-ahead distribution for  $y_t$ . Since these quantities are computed as part of the KF, the log likelihood element, or the two pieces of it separately, are usually provided, along with the filtered  $\mu_t, \Sigma_t$ , as part of the results of the filter.

## 8. THE KALMAN FILTER ESTIMATES STATES, NOT “PARAMETERS”

The KF assumes that  $A$ ,  $\Omega$ ,  $H$ , and  $\Xi$  are known quantities, while in applications they usually are not known. In applications in econometrics usually these parameters of the KF are specified as functions of some other underlying parameters, and the KF is executed to evaluate the posterior density at many values of the underlying parameters, either as part of a maximization routine or as part of a scheme for exploring the shape of the posterior density.

Since the KF assumes  $A$ ,  $\Omega$ ,  $H$ , and  $\Xi$  are known, and since the KF operates one date at a time, it can handle time subscripts on all these parameters. Of course if we have to estimate them, we don't want underlying parameters to be changing freely at every date, so the fact that the KF allows this is not of much help. However it is quite common in applications for  $H_t$  to be an observable matrix of exogenous variables that changes with  $t$ .

## 9. THE KALMAN SMOOTHER

- Kalman filter: the distribution for  $s_t \mid \mathcal{I}_t$  for each  $t$ . If we are making a decision in real time at each  $t$ , this is exactly what we need.
- But we are often looking at historical data to gain insight about parameter values and about past events. Then we want the distribution of  $s_t \mid \mathcal{I}_T$ , where  $T$  is the end of the sample.
- With the Kalman filter results in hand, this can be done recursively. The distribution of  $s_T \mid \mathcal{I}_T$  is delivered by the Kalman filter. The smoother at each  $t$  uses the distribution of  $s_{t+1} \mid \mathcal{I}_T$  and that of  $s_t \mid \mathcal{I}_t$ , to deliver that of  $s_t \mid \mathcal{I}_T$ .
- When we are done, we have a posterior distribution even for  $s_0 \mid \mathcal{I}_T$ , that will be different from the prior on  $s_0$ .

## 10. SMOOTHER FORMULAS

$$\begin{bmatrix} s_{t+1} \\ s_t \end{bmatrix} \mid \mathcal{I}_t \sim N \left( \begin{bmatrix} A\mu_t \\ \mu_t \end{bmatrix}, \begin{bmatrix} A\Sigma_t A' + \Omega & A\Sigma_t \\ \Sigma_t A' & \Sigma_t \end{bmatrix} \right).$$

Applying the formulas for normal conditional distributions,

$$(*) \quad s_t = \mu_t + \Sigma_t A' (A\Sigma_t A' + \Omega)^{-1} (s_{t+1} - A\mu_t) + \zeta_t,$$

where  $\zeta_t \sim N(0, \Sigma_t - \Sigma_t A' (A\Sigma_t A' + \Omega)^{-1} A\Sigma_t)$  and  $\zeta_t$  is uncorrelated with the past observations on  $y$  that generate  $\mathcal{I}_t$  and also with  $s_{t+1}$ . But the fact that it is uncorrelated with  $s_{t+1}$  means that it is necessarily also uncorrelated with  $s_{t+v+1}$  for all  $v > 0$ . This follows from the fact that the plant equation can be solved recursively to tell us that  $s_{t+v+1} = A^v s_{t+1} + \eta_{t+v+1}$ , where  $\eta_{t+v+1}$  is a linear combination of the plant error terms  $\varepsilon_{t+1+u}$  for  $u = 1, \dots, v$ . Since  $\zeta_t$  is uncorrelated with  $s_{t+1}$ , and

since  $\varepsilon_{t+u+1}$  is uncorrelated with any  $s_r$  or  $y_r$  for  $r \leq t + 1$ ,  $\zeta_t$ , a function of  $s_t, s_{t+1}$  and  $y_v, v \leq t$ , is uncorrelated with  $s_{t+1+v}$  for  $v \geq 1$ .

With this reasoning in hand, we can see that all we need do now is replace  $s_{t+1}$  in (\*) by its conditional distribution given  $\mathcal{I}_T$ , since everything else on the right-hand side is in  $\mathcal{I}_t$ . If we use the notation  $\{s_t \mid \mathcal{I}_T\} \sim N(m_t, S_t)$ , we therefore have

$$\begin{aligned} m_t &= \mu_t + \Sigma_t A' (A \Sigma_t A' + \Omega)^{-1} (m_{t+1} - A \mu_t) \\ S_t &= \Sigma_t - \Sigma_t A' (A \Sigma_t A' + \Omega)^{-1} A \Sigma_t \\ &\quad + \Sigma_t A' (A \Sigma_t A' + \Omega)^{-1} S_{t+1} (A \Sigma_t A' + \Omega)^{-1} A \Sigma_t. \end{aligned}$$

Here as with the formulas for the filter, you don't need to commit the formulas to memory. You should understand how they are derived.

## 11. FILTERING VS. SMOOTHING IN PRACTICE

Even if the underlying state is not changing at all, the filtered estimate of it will generally change a lot toward the beginning of the sample. Usually the prior is not very informative, so the distribution for the state changes strongly in response to the arrival of the initial data, then settles down as the sample lengthens. If this is happening, the smoothed estimates of the state will not show the rapid changes at the start of the sample, since they are using the full information set to estimate the state at every date.

If changes in  $s_t$  are the focus of interest, therefore, it is very important to use smoothed estimates of the state in interpreting changes in the estimated state. Also, even for the smoothed states, changes reflect both estimation error and actual movement in the state. Plots of smoothed estimates of the state should show standard error bands around the estimates, to give an idea of how much of the plotted movement might be due to estimation error.

## 12. APPLICATION: INDEX NUMBERS

Suppose we have a collection of  $N$  price time series  $p_{it}$ , measured in logs, that we think are each made up of an unobservable general "price level" component and an idiosyncratic component that is independent of the general price level and of other idiosyncratic components. We would like to use them to estimate the general price level. The equations are

$$\begin{aligned} p_{it} &= \alpha_i + \beta_i \bar{p}_t + v_{it} \\ \bar{p}_t &= \gamma_0 + \theta_0 \bar{p}_{t-1} + \varepsilon_{0t} \\ v_{it} &= \gamma_i + \theta_i v_{i,t-1} + \varepsilon_{it} \end{aligned}$$

$\varepsilon$  parameters are i.i.d. across time. They are independent across equations and independent of all lagged variables. Their variances,  $\sigma_i^2$ , may differ across equations.

To set this up as a KF problem, take the state to be  $\bar{p}_t$  together with  $v_{it}$ ,  $i = 0, \dots, N$  and a constant. We can either use a single constant that is always 1, or else treat the  $\alpha$  and  $\gamma$  coefficients as part of the state. The former keeps the KF simpler. The latter exploits the KF's ability to deliver posterior means for these constant term parameters, conditional on the other parameters, and thereby makes the dimension of an iterative posterior density maximization problem smaller.

The measurement equation is the first listed. The plant equation consists of the next two, plus either a single equation stating  $u_t = u_{t-1}$ , with the prior on  $u_0$  degenerate at  $u_0 = 1$ , or else  $2N+1$  equations of the form  $\alpha_{it} = \alpha_{i,t-1}$  and  $\gamma_{it} = \gamma_{i,t-1}$ .

*Practice exercise:* Figure out what the  $A$ ,  $H$ ,  $\Omega$  and  $\Xi$  matrices are in this problem, for both ways of treating the constant terms.

### 13. APPLICATION: TIME-VARYING PARAMETER REGRESSION

The model is

$$\begin{aligned} y_t &= X_t \beta_t + \varepsilon_t \\ \beta_t &= A \beta_{t-1} + v_t. \end{aligned}$$

The error terms  $\varepsilon_t$  and  $v_t$  are uncorrelated across equations and across time and are uncorrelated with  $X_t$  and with any lagged variables. Here the state is  $\beta_t$ , the first equation is the observation equation, and the second equation is the plant equation. We assume  $y_t$  and  $X_t$  are observable,  $\beta_t$  and the error terms are not.

*Practice exercise:* Figure out what the  $A$ ,  $H$ ,  $\Omega$  and  $\Xi$  matrices are in this problem. For the validity of the Kalman Filter, does it matter whether the  $X$ 's are strictly exogenous or instead predetermined? [Reminder: For strictly exogenous  $X$ 's,  $X_t$  and  $\varepsilon_s$  are uncorrelated for all  $t, s$  combinations, while for predetermined  $X$ 's  $X_t$  and  $\varepsilon_s$  are uncorrelated for  $s \geq t$ , but not necessarily for other  $t, s$  pairs. Predetermined  $X$ 's can be lagged  $y$ 's.]

### 14. APPLICATION: A FINITE ORDER MA MODEL

Suppose  $y_t = a_0 \varepsilon_t + a_1 \varepsilon_{t-1}$  with  $\varepsilon_t$  i.i.d.  $N(0, 1)$ . If we let  $[\varepsilon_t, \varepsilon_{t-1}]$  be the state vector, Then this equation becomes the observation equation and the plant equation is

$$\begin{bmatrix} \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{t-1} \\ \varepsilon_{t-2} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \end{bmatrix}.$$

Here of course  $v_t$  is the same thing as  $\varepsilon_t$ ; we only distinguish them to make the notation line up with that of the KF.

*Practice exercise:* Define the state and set up the KF plant and observation equations for an ARMA(1,1) model (i.e. a model of the form  $B(L)y_t = A(L)\varepsilon_t$ , with  $B$  and  $A$  both first-order polynomials). Can we treat any of the parameters in this model as part of the state?