ANSWERS TO PRACTICE PROBLEMS

(1) Suppose $y_t = .7y_{t-1} + \varepsilon_t$, $z_t = .9z_{t-1} + \nu_t$, where ε_t and ν_t are i.i.d. $N(0, I)$ and are uncorrelated with all y_s, z_s for $s < t$. (This makes them the innovations in the joint y, z process, of course.) Find the fundameental univariate MA representation for $x_t = y_t + z_t$. [Hint: This will take the form $(P(L)/Q(L))\eta_t$, where η_t is the univariate innovation in x_t. Form the acf of x, as a ratio of polynomials in the lag operator, and then factor to get expressions in positive powers of L with no roots inside the unit circle.]

The acf is the sum of the two individual process acf's, i.e., as coefficients in a polynomial in L,

$$\frac{1}{(1-.7L)(1-.7L^{-1})} + \frac{1}{(1-.9L)(1-.9L^{-1})}$$
$$= \frac{3.3 - 1.6(L + L^{-1}}{(1.81 - .9(L + L^{-1})(1.49 - .7(L + L^{-1})}\,.$$

To find the fundamental representation, we have to factor this into pieces, one of which involves only non-negative powers of L and has no zeroes inside the unit circle, the other of which has the opposite properties. The denominator of the polynomial is pre-factored for us. The numerator needs to be represented as

$$\alpha_0^2 + \alpha_1^2 + \alpha_1(L + L^{-1}) = (\alpha_0 + \alpha_1 L)(\alpha_0 + \alpha_1 L^{-1})\,,$$

with, to guarantee fundamentalness, $|\alpha_0| > |\alpha_1|$. *My calculations suggest* $\alpha_0 = 1.433$, $\alpha_1 = 1.117$ *is the answer.*

(2)

$$p_{it} = \alpha_i + \beta_i \bar{p}_t + \nu_{it}$$
$$\bar{p}_t = \gamma_0 + \theta_0 \bar{p}_{t-1} + \varepsilon_{0t}$$
$$\nu_{it} = \gamma_i + \theta_i p_{i,t-1} + \varepsilon_{i,t-1}$$

ε parameters are i.i.d. across time. They are independent across equations and independent of all lagged variables. Their variances, σ_i^2 , may differ across equations.

Figure out what the A, H, Ω and Ξ matrices are in this problem, for both ways of treating the constant terms.

It should have been a ν_{t-1} *rather than an* ε_{t-1} *in the third equaion. As written, the problem is harder and doesn't make much sense. However, taking it as given,*

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and using a single constant-term, the state vector is $(\vec{p}, \bar{p}, \vec{\varepsilon}, 1)$ and we have

$$A = \begin{bmatrix} \vec{\theta} & \beta & I & \vec{\alpha} + \vec{\gamma} \\ 0 & \theta_0 & 0 & \gamma_0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix},$$

$$\Omega = \begin{bmatrix} \operatorname{diag}(\vec{\sigma}^2) & 0 & \operatorname{diag}(\vec{\sigma}^2) & 0 \\ 0 & \sigma_0^2 & 0 & 0 \\ \operatorname{diag}(\vec{\sigma}^2) & 0 & \operatorname{diag}(\vec{\sigma}^2) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \Xi = 0,$$

where we use the $\vec{}$ notation to indicate a vector whose elements are indexed by $i = 1, \ldots, N$.

For the second way of setting this up, we replace the 1 at the end of the state vector with $(\vec{\alpha}, \gamma_0, \vec{\gamma})$. Then we have

$$A = \begin{bmatrix} \vec{\theta} & \beta & I & I & 0 & I \\ 0 & \theta_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

The H, Ω and Ξ matrices remain the same, except that some of the zeros in them now stand for bigger matrices.

(3)

$$y_t = X_t\beta_t + \varepsilon_t$$
$$\beta_t = A\beta_{t-1} + \nu_t\,.$$

Figure out what the A, H, Ω and Ξ matrices are in this problem. For the validity of the Kalman Filter, does it matter whether the X's are strictly exogenous or instead predetermined?

Here the state vector is β_t and the second equation is the plant equation, with its A the plant equation A. We have a time-varying $H_t = X_t$. Ω and Ξ are the variances of ν_t and ε_t. The Kalman filter uses the conditional distribution of y_{t+1} given information at t. The predeterminedness assumption on X_t is enough to deliver the assumptions of the Kalman filter. Strict exogeneity is not required.

(4) Define the state and set up the KF plant and observation equations for an ARMA(1,1) model (i.e. a model of the form $B(L)y_t = A(L)\varepsilon_t$, with B and A both first-order polynomials). Can we treat any of the parameters in this model as part of the state?

The model will then be $(1 - bL)y_t = (1 - aL)\varepsilon_t$*. Let the state be* y_t, ε_t*. Then the plant equation is*

$$\begin{bmatrix} y_t \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} b & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \zeta_t \\ \zeta_t \end{bmatrix} .$$

and the H *matrix is* $[1 \;\; 0]$*. Since* y_{t-1} *is observed, we can move it over into a time-varying* A_t *and treat* b *as part of the state vector. Then the plant equation becomes*

$$\begin{bmatrix} y_t \\ \varepsilon_t \\ b_t \end{bmatrix} = \begin{bmatrix} 0 & a & y_{t-1} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \varepsilon_{t-1} \\ b_{t-1} \end{bmatrix} + \begin{bmatrix} \zeta_t \\ \zeta_t \\ 0 \end{bmatrix}$$

We also have $\Xi = 0$*,* $H = [1\ 0\ 0]$*, and*

$$\Omega = \begin{bmatrix} \sigma_\zeta^2 & \sigma_\zeta^2 & 0 \\ \sigma_\zeta^2 & \sigma_\zeta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$