MID-TERM EXAM

There are 5 questions, including one on a second page. Answer all 5 questions. There are 90 points in total, including 5 bonus points awarded free to everyone who hands in the exam.

(1) (15 points) Suppose the Gaussian process X_t satisfies

$$\mathrm{Cov}(X_t, X_{t-s}) = \begin{cases} 1 & s \text{ even} \\ 0 & s \text{ odd} \end{cases} .$$

(a) Is this process stationary?
If the mean is constant, it is stationary, as the covariances between X's at different dates depend only on their separation in time, not their absolute location. Since we are given that this is a Gaussian process, means and covariances completely characterize joint distributions, so this is all we need to check.

(b) Is this process linearly regular? If so, display its moving average representation.
No. Because the correlation of X_t with X_{t-s} is 1 for even s, X can be predicted perfectly arbitrarily far into the future, and is therefore indeed linearly deterministic.

(c) Is this process linearly deterministic? If so, determine $E[X_t \mid \{X_{t-v},\, v \geq 99\}]$.
Assuming mean zero, the answer is simply X_{t-100}. Though X_{t-99} is in the information set, since it is an odd lag, it is uncorrelated with $X(t)$ and therefore no help in forecasting. But because of the perfect correlation, $X_t = X_{t-100}$.

(d) Is this process ergodic? Prove your answer is correct.
No. Any given realization of this process will oscillate between an "even" value of X_t and an "odd" value. It will be exactly periodic with period 2. So a time average of X over one realization will converge to $(X_1 + X_2)/2$, which is a random variable, not $E[X_t] = 0$.

(2) (15 points) Consider the function

$$S_X(\omega) = e^{-(\pi-\omega)^{-1}(\pi+\omega)^{-1}}, \quad \omega \in [-\pi, \pi] .$$

(a) Sketch the shape of this spectral density function. Is it the spectral density of a stationary Gaussian process? How do you know?
It's everywhere positive, symmetric around 0, and integrable. Therefore it is the spectral density of a stationary Gaussian process. (We can define a process in the frequency domain with independent increments and variances of its

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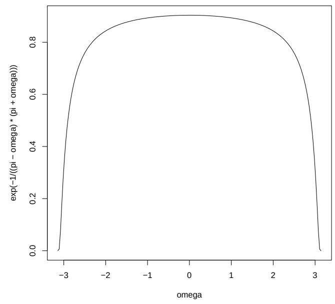
integrals over intervals defined by this spectral density. It will then have an inverse FT that is a stationary, finite-variance process.). A plot:



(b) Is it the spectral density of a linearly regular process? How do you know?
No, because a linearly regular process has to have a spectral density such that

$$\int_{-\pi}^{\pi} \log S(\omega)\, d\omega > -\infty\,.$$

Here that amounts to

$$\int_{-\pi}^{\pi} \frac{-1}{(\pi-\omega)(\pi+\omega)} d\omega > -\infty\,.$$

But in the neighborhood of π and $-\pi$, this integral behaves like $\int_0^a (1/x)\,dx$, which is infinite, so the condition is not met. Note, though, that because the spectral density is close to flat over most of $(-\pi,\pi)$, realizations of the process will look a lot like those of an i.i.d. process. It is only with a very long sample that the possibility of predicting with arbitrary precision would become apparent.

(3) (15 points) $y_t = \varepsilon_t + .7\varepsilon_{t-1} - .4\varepsilon_{t-2}$, where ε process is i.i.d. $N(0,1)$.

(a) What is the variance of y_t conditional on knowledge of $\{\varepsilon_s,\ s<t\}$?
This prediction error is just ε_t, so its variance is 1.0.

(b) What is the variance of y_t conditional on knowledge of $\{y_s,\ s<t\}$?
The roots of the MA polymial here are

$$\frac{-.7 \pm \sqrt{.49+1.6}}{.8} = (1/1.0728, -1/.3728)\,,$$

one of which is inside the unit circle, so projecting on past ε's is not the same as projecting on past y's. So we have to "flip a root". The polynomial can be written as

$$(1+.3728L)(1-1.0728L) = -1.0728L(1+.3728L)(1-(1/1.0728)L^{-1})$$

We can get a new MA operator that implies the same ACF as the original by replacing the L^{-1} with an L in this expression. The lead term in the new MA operator will be 1.0728, which reflects the scaling up in the variance of the *innovation that comes from using only past y's, instead of past ε's, in forming forecasts. So the new forecast error variance is* $1.0728^2 = 1.1509$.

(4) (25 points) y_t's fundamental MA representation is $y_t = \varepsilon_t + \alpha\varepsilon_{t-1}$.

(a) Write out the plant and observation equations for a Kalman filter, treating $s_t = (\varepsilon_t, \varepsilon_{t-1})$ as the state and α as known.

$$\begin{aligned} \textit{plant:} \quad & s_t = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} s_{t-1} + \begin{bmatrix} \varepsilon_t \\ 0 \end{bmatrix} \\ \textit{observation:} \quad & y_t = \begin{bmatrix} 1 & \alpha \end{bmatrix} s_t \end{aligned}$$

(b) With $y_1 = 1$, $y_2 = 0$ being the whole sample, calculate filtered and smoothed estimates of the state at $t = 1, 2$, assuming $\alpha = .7$ and that the initial prior covariance matrix for s_1 is the identity. Also calculate the log marginal data density (likelihood times prior) for this value of α. 2-decimal place accuracy is enough, and for the likelihood you can leave logs unevaluated, if you're not doing this on a calculator.

The prior on s_1 *is* $N(0, 1)$*, and we observe* $[1\ \alpha]s_1$*. (We could have more conventionally said that we had a* $N(0, 1)$ *prior on* s_0*, which would have implied this same pre-observation belief about the distribution of* s_1*.) The distribution of* $s_1 \mid y_1$ *is then determined by*

$$E_1[s_1] = ((y_1 - E_0 y_1)\operatorname{Var}_0(y_1)^{-1}\operatorname{Cov}_0(y_1, s_1) = 1 \cdot (1 + \alpha^2)^{-1} \begin{bmatrix} 1 \\ \alpha \end{bmatrix} = \begin{bmatrix} .6711 \\ .4698 \end{bmatrix}$$

$$\operatorname{Var}_1(s_1) = I - \begin{bmatrix} .6711 \\ .4698 \end{bmatrix} [1 \quad .7] = \begin{bmatrix} .3289 & -.4698 \\ -.4698 & .6711 \end{bmatrix} .$$

The log likelihood element for this first observation is $-\frac{1}{2}\log 1.49 - \frac{1}{2} 1/1.49 = -.5350$.

Conditional on this first observation, the distribution for s_2 *has mean* $E_1 s_2 = [0 \quad .6711]'$ *and*

$$\operatorname{Var}_1(s_2) = \begin{bmatrix} 1 & 0 \\ 0 & .3289 \end{bmatrix} .$$

The observation y_2 *has* $E_1 y_2 = .4698$ *and* $\operatorname{Var}_1(y_2) = 1.1612$*. Again applying the usual Gaussian regression formulas, we find that*

$$E_2 s_2 = \begin{bmatrix} -.4046 \\ .5780 \end{bmatrix}$$

$$\operatorname{Var}_2(s_2) = \begin{bmatrix} .8618 & .1982 \\ .1982 & .0456 \end{bmatrix}$$

The log likelihood element for this observation is

$$-\frac{1}{2}(\log(1.1612) + (-.4698)^2/1.1612) = -.1698$$

So we're done with the filtering, and adding up the likelihood elements will give us the marginal likelihood. The smoothed estimate of ε_1 is already available as the filtered $t = 2$ estimate of the second state variable. All that remains is the smoothed estimate of ε_0. But we already know exactly, at time t=1, what $\varepsilon_1 + .7\varepsilon_0$ is — it's 1.0, the observed value of y_1. So we get the smoothed estimate of ε_0 by solving $.5780 + .7\hat{\varepsilon}_0 = 1$, i.e. $\hat{\varepsilon}_0 = .6029$.

(5) (15 points) Though there is no unique optimum deseasonalizing filter, such a filter ought to remove power in a narrow band about each seasonal frequency and change the series it is applied to as little as possible at non-seasonal frequencies. With these qualities in mind, find the Fourier transforms of each of the filters below and discuss how well each one meets the criteria for a deseasonalizing filter for monthly data.

(a) $\frac{1-L^{12}}{1-L} = \sum_{s=0}^{11} L^s$

The FT of this is $(1 - \exp(-12i\omega))/(1 - \exp(-i\omega))$. Its absolute value is $(2 - 2\cos(12\omega))/(2 - 2\cos(\omega))$. It is easy to see that this has zeros at $2\pi j/12$, for integer $j \neq 0$, and an application of l'Hôpital's rule tells us that it takes on the value 12 at 0. The 12 at zero means that at that frequency variation is scaled up by a factor of 12, so we might consider dividing by 12. But at other non-seasonal frequencies above $\pi/6$ the peaks are far smaller. So this filter meets the criterion of being small at seasonal frequencies, but it substantially distorts the relative power at different non-seasonal frequencies. It could be useful if divided by 12 and if we were interested only in non-seasonal variation at frequencies lower than $\pi/6$, i.e. periods longer than a year. Note that because the filter is not symmetric about zero, its Fourier transform is complex, meaning it "phase shifts" the data. The phase shift is zero to first order in the neighborhood of the zero frequency, but non-zero elsewhere. Roughly speaking, since data at t are replaced by an average over the year preceding t, a delay of six months is introduced.

(b) $1 - \sum_{s=-3}^{3} L^{12s}/7$

This FT is

$$1 - \frac{1}{7}\frac{e^{36i\omega} - e^{-48i\omega}}{1 - e^{-12i\omega}} = 1 - \frac{1}{7}\frac{\sin(42\omega)}{\sin(6\omega)}\,.$$

L'Hôpital's rule again tells us that this is 0 at $2\pi j/12$, this time for all integer j, including $j = 0$. The slope gets steep fairly quickly as we move away from seasonal frequencies, meaning that a fairly narrow band of power is removed, and the filter is periodic, so non-seasonals are treated the same, whether they are high or low frequencies. The FT is real, so there is no phase shift. The only *real problem is the elimination of all power at 0, which is not really a seasonal frequency.*

(c) $1 - \sum_{s=-3}^{3} L^{4s}/7$
This one has FT $1 - \sin(28\omega)/(7\sin(4\omega))$. *It has zeros at* $2\pi j/4$*, so it misses quite a few seasonal frequencies. It would be OK for quarterly data, not for monthly.*

(d) $1 - \sum_{s=0}^{3} L^{12s}/4$
This has a shorter moving average than that in 5b, so it will take out variation in a less narrow band. Also, it's asymmetric, so it will induce phase shift. So on the whole it is worse than 5b, though it shares some of its characteristics.