ANSWERS FOR EXERCISE DUE MONDAY, 10/3

(1) The probability space is $S = \{1,2,3,4,5\}$. The probability of every point ω in S is $1/5$. We define random variables X_i by

$$
\begin{aligned}
X_1(5) &= 2 & & \\
X_1(\omega) &= 1 & & \omega < 5 \\
X_2(1) &= 2 & & \\
X_2(\omega) &= 1 & & \omega > 1 \\
X_3(3) &= 2 & & \\
X_3(\omega) &= 1 & & \omega \neq 3\,.
\end{aligned}
$$

Let $\mathcal{F}_t$, $t = 1, \dots, 3$ be defined as the σ-field generated by X_s, $s \leq t$.

(a) Display the sets making up each of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$.

$\mathcal{F}_1$: $\emptyset, \{5\}, \{1,2,3,4\}, \{1,2,3,4,5\}$
$\mathcal{F}_2$: $\mathcal{F}_1 \cup \{1\} \cup \{2,3,4,5\}, \cup \{2,3,4\}, \cup \{1,5\}$
$\mathcal{F}_3$: *all unions of the sets* $\emptyset, \{1\}, \{2,4\}, \{3\}, \{5\}$

(b) Could these three random variables form part of a stationary process?

There are two approaches to answering this. We can put the probability space S aside and just look at the joint distributions. We note then that each X_i has probability .2 of being equal to 2, probability .8 of being equal to 1. So all the univariate distributions are the same. We also need that the two pairwise joint distributions of adjacent X_i's are the same. But these also match, with probability .2 on (2,1), .2 on (1,2), and .6 on (1,1). So these joint distributions are translation-invariant.

A slightly different question is whether on this same probability space $(S, \mathcal{F}, P)$ we could define a stationary process on $\mathbb{R}^{\mathcal{Z}}$ with these three random variables as X_1, X_2, X_3. One way to do that is to say that $X_{3s+1}(5) = 2$ for all integer s, while $X_{3s}(5) = X_{3s+2}(5) = 1$ for all integer s. So for $\omega = 5$, the X_s path is periodic, of the form $2,1,1,2,1,1,2,1,1,\dots$. Similarly for $\omega = 1$, we make $X_t(\omega)$ periodic, but starting at 1, so its pattern is $1,2,1,1,2,1,1,2,1,\dots$, and for $\omega = 3$ the pattern is $1,1,2,1,1,2,1,1,2,\dots$. Then for $\omega = 2$ or $\omega = 4$, $X_t(\omega) \equiv 1$. This defines a stationary process, because there are only four possible time paths for X. The constant time path of course is invariant with time shifts. The three paths that alternate 2's and 1's all have the same probability, and they translate into one another under time shifts. Every event defined in terms of X_t values will be some collection of these four possible time paths, and hence will have a probability that is invariant under time shifts.

(c) Find $\mathrm{Cov}(X_i, X_j)$ for all combinations of $i, j = 1, \dots, 3$.

Date: October 3, 2005.

$\mathrm{Cov}(X_i, X_i) = 1.6 - 1.2^2 = .16$, $t = i, \ldots, 3$. $\mathrm{Cov}(X_i, X_j) = 1.4 - 1.44 = -.04$, $i \neq j$.

(d) Find $E_t[X_3]$ and $\mathrm{Var}_t[X_3]$ for $t = 1, 2$, evaluated at $X_1 = 1, X_2 = 1$, at $X_1 = 2, X_2 = 2$, and at $X_1 = 2, X_2 = 1$. Note that these variables are not joint normal; the conditional expectations will not be linear functions.
$E[X_3 \mid X_1 = 1] = 1.25$, $E[X_3 \mid X_1 = 2] = 1$, $E[X_3 \mid X_1 = 1, X_2 = 1] = 4/3$, $E[X_3 \mid X_1 = 2, X_2 = 2]$ *undefined (conditioning on set of measure zero)*, $E[X_3 \mid X_1 = 2, X_2 = 1] = 1$.

$$\mathrm{Var}[X_3 \mid X_1 = 1] = .0375$$
$$\mathrm{Var}[X_3 \mid X_1 = 2] = 0$$
$$\mathrm{Var}[X_3 \mid \{1,2\}] = \mathrm{Var}[X_3 \mid \{2,1\}] = 0$$
$$\mathrm{Var}[X_3 \mid \{1,1\}] = .2222$$

Note that in a jointly Gaussian process, conditional variances can never be greater than unconditional variances, but in this non-Gaussian case, conditioning on $X_1 = 1, X_2 = 1$ yields a conditional variance greater than the unconditional variance.

(2) (a) For each of the sets of moving average weights a below, compute and plot the acf of $X_t = \sum a_i \varepsilon_{t-i}$ for time separations $s = -15, \ldots, 15$. This will be tedious unless you use the computer.

(b) For each of the sets of moving average weights a below, compute and plot 5 simulated draws for X_t, $t = 1, \ldots, 50$ by generating 60 i.i.d. $N(0, 1)$ random draws and averaging them with a. Note that you can draw a single set of 5 i.i.d. ε sequences and use the same 5 for each of the a's. This makes it clearer what differences are due to the a's alone. All 5 lines for a single a should be on the same plot.

The a's:

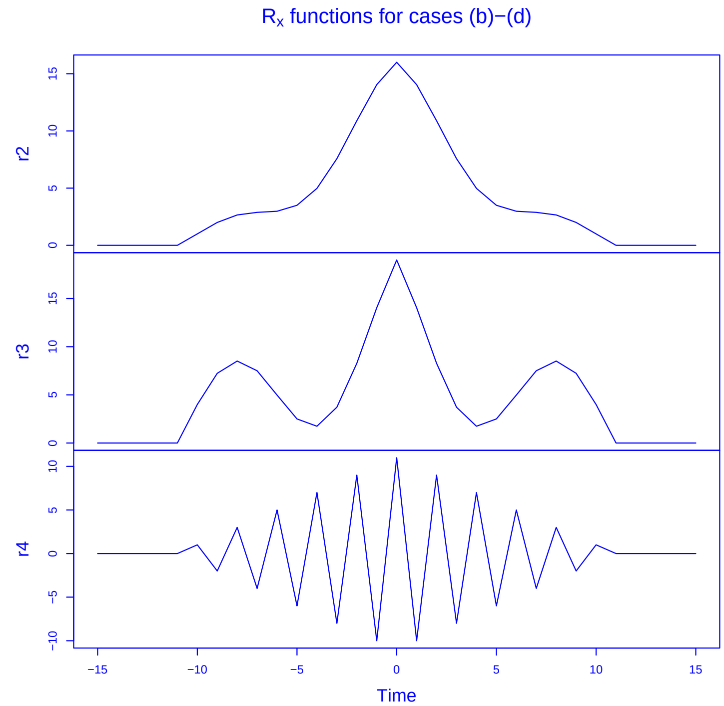
(a) $a_i = 1, i = 0, \ldots, 10$
(b) $a_i = \sin(2\pi i/10) + 1, i = 0, \ldots, 10$
(c) $a_i = \cos(2\pi i/10) + 1, i = 0, \ldots, 10$
(d) $a_i = (-1)^i, i = 0, \ldots, 10$

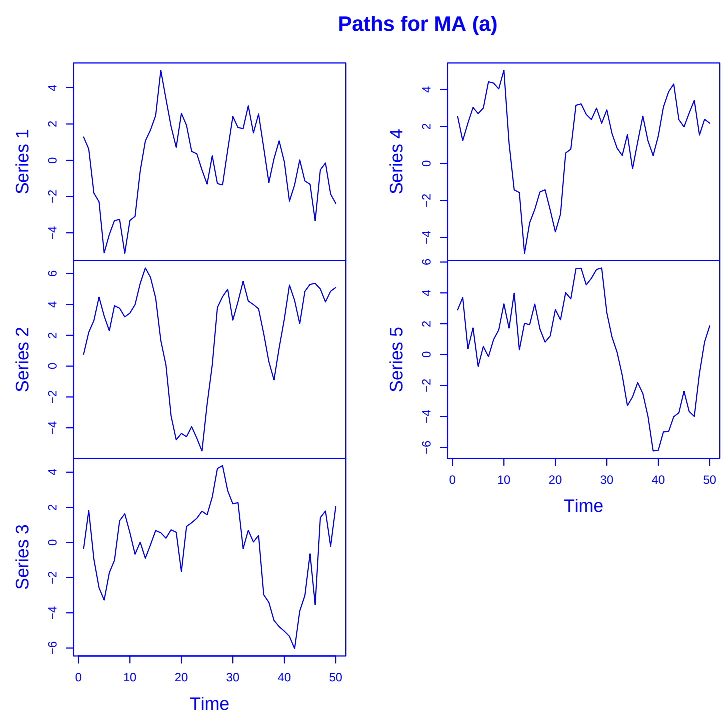
Command to generate a 60 by 5 matrix of $N(0, 1)$ random variables:

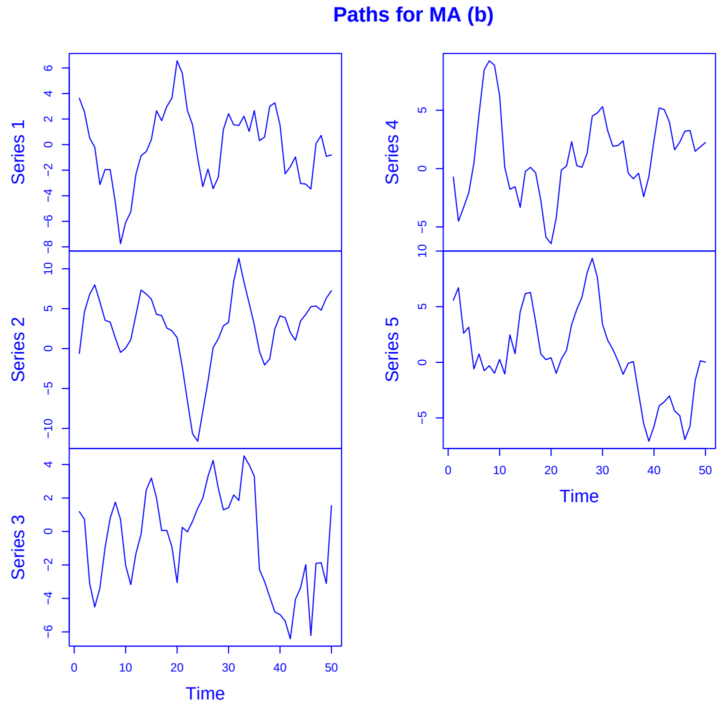
matlab: `z = nrand(60,5)`

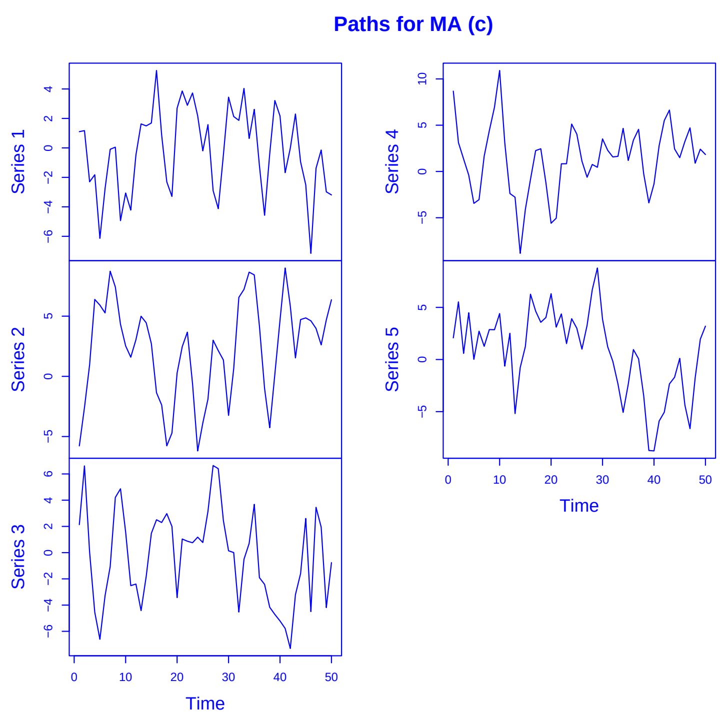
R: `z <- matrix(rnorm(60*5),ncol=5)`

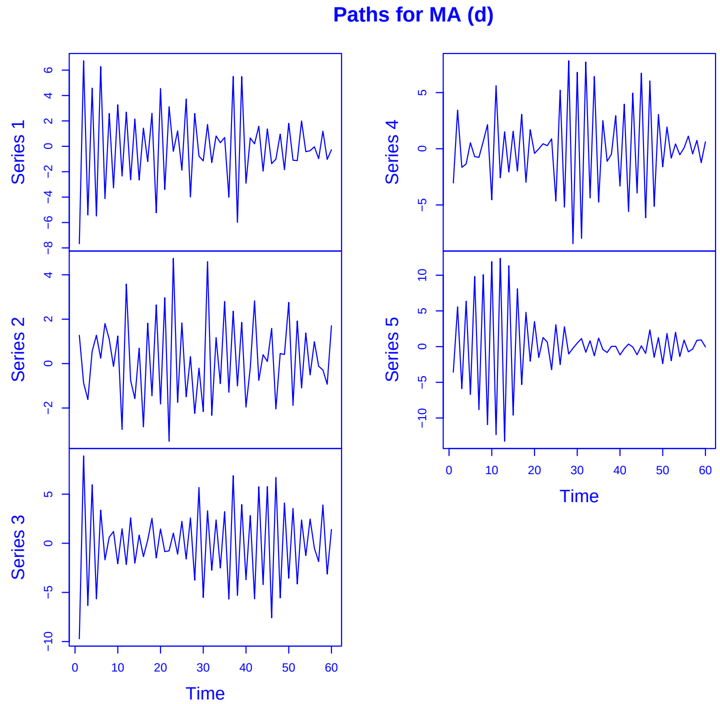
I decided that it was actually clearer to show the draws of separate path realizations for a given process on separate small graphs. Here they are, as well as graphs of the three non-trivial R_X functions.


R_x functions for cases (b)−(d)
r2
r3
r4
Time


Paths for MA (a)
Series 1
Series 2
Series 3
Series 4
Series 5
Time
Time


Paths for MA (b)
Series 1
Series 2
Series 3
Series 4
Series 5
Time
Time


Paths for MA (c)
Series 1
Series 2
Series 3
Series 4
Series 5
Time
Time



Note that the two trigonometric weight functions produce paths with a tendency to oscillate with period 10, and that the $(-1)^t$ *weights produce paths that oscillate with period 2.*

(3) There are, or will be, monthly data on the Federal Funds rate on the course web site. Using these data, find a maximum likelihood estimate of the weights a in a 12th-order Gaussian MA model with constant mean $\bar{r}$ for these data. Determine whether the the MA weights to which your estimates have converged are fundamental. [You can use "root-flipping", which we will probably cover in the 9/28 lecture, or you can try starting the maximization from a different place to get convergence to a different set of weights, so you can compare a_0's, or you can construct an approximation to the one-step-ahead predictor by using a large finite number of lags and see if its residual variance is close to a_0^2.]

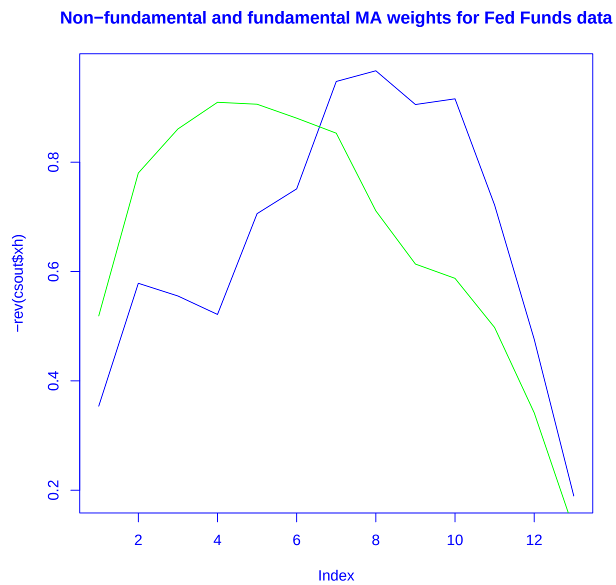
I found convergence to weights:

-0.1894538 -0.4761951 -0.7215926 -0.9159184 -0.9054663 -0.9672040 -0.9476629 -0.7512126 -0.7058590 -0.5214969 -0.5552680 -0.5784963 -0.3537538

These are obviously not fundamental, if the first is interpreted as a_0*, but they do not generate a fundamental MA representation even if the weights are taken in reverse order. The fundamental weights, which I verified both using the cholesky decomposition method and root flipping, are:*

0.5188256 0.7802005 0.8607574 0.9095844 0.9060185 0.8805265 0.8530326 0.7110018 0.6136318 0.5875334 0.4976141 0.3416769 0.1291763

The weights are plotted below. Note that the fundamental weights (the green line) are shifted to the left relative to the non-fundamental ones. Engineers call the fundamental representation the "minimum delay" representation, because in a certain sense it has the maximal concentration of weights near zero.



The MLE of the mean is 5.6682.

Note that this is quite a bad model for this series. The sample acf for the data shows an autocorrelation of about .3 even at 60 months, and then a turn to a negative autocorrelation of about .3 at 200 months. This MA(12) model obviously implies that all the autocorrelations beyond 12 months are zero.