Econ. 513, Time Series Econometrics Fall 2001 Chris Sims

# Exercises due Tuesday, 11/27*

(1) Consider the simplest possible case of Monte Carlo methods. We wish to draw from a target pdf $\pi$ for the random variable $X$ on the discrete space $\{1,2\}$. The target pdf is determined by $\pi_1 = \frac{1}{3}$. We would like to know the value of $E[X^2]$, which of course we could easily calculate analytically as 3. But instead we will do it the hard way.

(a) By importance sampling: We draw $N$ i.i.d. values for $X$ independently from the pdf $q$ that makes $q_1 = \frac{1}{2}$ and weight them appropriately so that averaging the weighted draws will allow us to calculate $E[X^2]$. What will the weights be? Calculate the pdf of the weighted average of 4 draws of $X^2$ from this distribution and plot it. (It will be concentrated on just 5 possible values, according to the counts of 1's and 2's drawn.) Calculate the variance of the distribution. Compare the distribution to that of the sample average of 5 draws of $X^2$ drawn directly from the target distribution.

(b) By Metropolis sampling: What family of jump distributions $p(j\,|\,i)$ satisfies the symmetry requirement of the Metropolis algorithm? Assuming a jump distribution with $p(j\,|\,i) = .5$ and that the initial draw is from the target unconditional pdf with $\pi_1 = \frac{1}{3}$, what is the pdf of the average of 4 draws by the Metropolis algorithm? Compare to the distribution of the estimator from the previous part based on importance sampling.

[This exercise involves combinatorics and the binomial distribution, which you may or may not be experienced with. The probability of getting $n$ draws of 1 out of $m$ i.i.d. draws in which the probability of 1 is $p$ at each draw is given by

$$\binom{n}{m} p^n (1-p)^{m-n} = \frac{m!}{n!(m-n)!} p^n (1-p)^{m-n} \,.$$

Matlab has a function `nchoosek.m` that calculates binomial coefficients (the ratios of factorials above) or lists of combinations. For the Metropolis sampling, which is of course not i.i.d., you probably need to separately calculate the probabilities of each of the 16 possible sequences of 1's and 2's.]

(2) Consider the model

$$y_t = c + \rho y_{t-1} + \varepsilon_t$$

with the usual assumption that $\varepsilon_t$ is independent of past $y$'s and $N(0,\sigma^2)$. Suppose we wish, using a flat prior over (-1,1) for $\rho$, an improper flat prior for $c$, a $\sigma^{-2}$ (improper) prior on $\sigma^2$, and the full unconditional likelihood (meaning that the model's implied unconditional pdf under stationarity is used for the initial $y$), to evaluate the joint posterior for $c$, $\rho$ and $\sigma^2$. Using quarterly data for the natural log of US real GDP, 1947-2001:3 as $y$.

(a) Plot the contours of the joint posterior pdf for $\rho$ and $c$. This will require calculating the joint posterior pdf for $c$, $\rho$ and $\sigma^2$ at a grid of points, then summing over the $\sigma^2$ dimension.

(b) Plot the marginal pdf's of $\rho$ and $c$.

(c) Approximate the posterior expectation of $\rho$ and $c$ by drawing a sample of 1000 from the posterior by using the flat-prior normal-inverse-gamma distribution obtained by conditioning on the initial conditions and weighting appropriately. (I.e., use importance sampling.) [Note: some draws will be outside [-1,1]. They will just have zero weight.]

(d) Draw an artificial sample of 1000 from the posterior by Metropolis sampling, using as a jump distribution a joint normal on $c$, $\rho$ and $\log\sigma^2$. Use the OLS estimates as a starting point. Make the jumps independent on the three parameters and make their variances one fourth of the variances of the OLS posterior. Here jumps to outside [-1,1] for $\rho$ are always rejected, because the true posterior pdf is zero there. Plot the resulting joint distribution for $c$ and $\rho$ as a scatter plot superimposed on a contour plot like that you found in part 2a.

(3) Consider the model

$$\begin{aligned} y_t &= c_t + (1-\rho L)^{-1}\varepsilon_t \\ c_t &= c_{t-1} + \nu_t\,, \end{aligned}$$

where $\nu$ and $\varepsilon$ are i.i.d. jointly normal and independent of each other, with variances $\sigma_\varepsilon^2$ and $\sigma_\nu^2$. $y_t$ is observed, $c_t$ is not.

(a) Show that this is an ARMA model, i.e. that its fundamental MA operator can be represented as the ratio of two finite order polynomials.

(b) Show how, taking the state vector to be $[y_t\; c_t]$, this model fits the Kalman Filter framework.

(c) Using data on the log of real GDP, quarterly over 1948:1-2001:3, estimate the parameters of this model ($\rho$, $\sigma_\varepsilon^2$, and $\sigma_\nu^2$) by maximizing the posterior pdf under a flat prior. You will need to assume some distribution for the presample values of $c$ and $y$. Take it to be

$$N\left(0, \begin{bmatrix} 1000 & 1000 \\ 1000 & 1000+\frac{\sigma_\varepsilon^2}{1-\rho^2} \end{bmatrix}\right)$$

for $\sigma_\varepsilon^2 < 1000(1-\rho^2)$, and with the lower right element of the covariance matrix 2000 otherwise. You should use the Kalman filter to evaluate likelihood and a nonlinear optimization routine (e.g. `csminwel.m`) to maximize the posterior.

(d) Compare the fit of this model to that of a simple second-order linear AR. Using the Schwarz criterion or some variant on it would be appropriate.