Econ. 513, Time Series Econometrics Fall 2001 Chris Sims

Notes and Exercise on Invertibility Criteria*

1. "LAST COEFFICIENT TOO BIG" IN MULTIVARIATE MODELS

As we discussed in class, it is clear that an n'th univariate polynomial $A(L)$ cannot have all its roots outside the unit circle if $|A_n| > |A_0|$. This is clear from the fact that

$$A(L) = A_0 \prod_{i=1}^{n} (1 - \rho_i^{-1} L) ,$$

where ρ_i is the i'th root of the polynomial $A(z)$ thought of as having complex argument z. In this representation we see that

$$A_n^{-1} = \prod_{i=1}^{n} \rho_i ,$$

so of course if $|A_n| > 1$, at least one of the roots must be inside the unit circle.

When $A(L)$ is a matrix polynomial, perhaps the easiest way to see the corresponding result is to note that in the $A_0 = I$ case the roots of the $|A(z)|$ polynomial are exactly the inverses of the eigenvalues of the coefficient matrix from the corresponding stacked first-order dynamic system:

$$\begin{bmatrix} A_1 & A_2 & \dots & A_{n-1} & A_n \\ & & I & & 0 \end{bmatrix} .$$

Because of the block structure of the matrix, it is clear that the determinant of the matrix is $|A_n|$. If its determinant exceeds one in absolute value, then at least one of the roots of $|A(z)|$ must lie inside the unit circle. To translate this to the case where $A_0 \neq I$, we get the condition that $|A_0| > |A_n|$ in absolute value. In other words, the univariate condition on the relative absolute sizes of A_n and A_0 translates into a condition on relative absolute sizes of determinants in the multivariate case.

We should emphasize again that these conditions are just very weak necessary conditions that are often useful as quick checks in practice. It is easy to construct examples of systems in which A_0 is bigger than A_n in every way, yet $|A(Z)|$ does not have all its roots outside the unit circle.

2. Finding the fundamental MA from a non-fundamental MA

This is not easy in a multivariate case. In a univariate case, if the non-fundamental MA operator is $a(L)$ and being non-invertible has a root z_0 inside the unit circle, we can write $a(L) = b(L)(1 - z_0^{-1}L)$. Then we can "flip" the root, replacing $a(L)$ with $a^*(L) = b(L)(1 - z_0 L)/z_0$. This will imply the same autocovariance function as the original a. If z_0 is complex a^* may come out complex, but because complex roots come in conjugate pairs, they always need to be flipped in pairs, and the final result will be real. Once all roots inside the unit circle have been flipped, we have the fundamental MA.

An analogous algorithm in a multivariate model is much harder, because of the non-commutativity of matrix multiplication. Flipping a root is always possible, however, by the following algorithm.

Suppose $A(z)$ has all its elements finite order polynomials in z and $|A(z_0)| = 0$. Then we can take the singular value decomposition of $A(z)$ at the root, to obtain $A(z_0) = UDV'$, where $U'U = V'V = I$ and D is diagonal with non-negative elements. At least one of the diagonal elements of D must be zero, and we can arrange it so that that element appears as, say d_n, the last diagonal element of D. Note that, in case matrices have complex elements, we are interpreting "X'" to imply both transposition of X and complex conjugation of its elements. Then

$$B(L) = A(L)V \begin{bmatrix} I & 0 \\ 0 & \dfrac{1 - z_0 z}{z_0(1 - z_0/z)} \end{bmatrix}, \tag{1}$$

has all its elements finite-order polynomials in L and obviously has replaced (one of) A's z_0 root(s) with a $1/z_0$ root. That the division by $1 - z_0/z$ does not make any elements of the B fail to be finite-order follows from the fact that, by construction, all the elements in the last column of $A(z)V$ vanish at $z = z_0$, and thus are polynomials with a root at z_0. Finally, we can observe that $B(L)B'(L^{-1}) = A(L)A'(L^{-1})$, so that B and A define the same autocovariance function. Repeating this operation for every root that is on the wrong side of the unit circle, we can arrive at the fundamental representation. Note that it is important in applying this algorithm that if we begin with an MA representation $y = C(L)\varepsilon$ with $\mathrm{Var}(\varepsilon) = \Sigma \neq I$, we factor Σ as $W'W$ and use $y = C(L)W'\eta = A(L)\eta$ as the moving average representation, so that $R_y(L) = A(L)A'(L^{-1})$.

This algorithm, when some roots are complex, may deliver complex coefficients for $B(L)$, even after both elements of a conjugate pair have been "flipped". If the original A was real, however, B can be made real by post-multiplying it by the unitary matrix $B_0'W^{-1}$, where $W'W = B_0B_0'$ and W is real. W can be found, e.g., by Choleski decomposition.

A computer program that executes this algorithm is now on the course web site, so that the second exercise, originally “extra credit”, becomes straightforward using this program.

3. Exercise due Thursday, 10/25

The first problem should be very easy, though you may need a computer. The second is extra credit — you don't have to do it.

(1) Determine, as fast as possible, which of the following are fundamental MA operators. State in each case how you reached your conclusion.
   (a) $1 + 2L + 3L^2 + 4L^3$
   (b) $1 + 2L + 3L^2 + 2L^3 + L^4$
   (c) $I + \begin{bmatrix} .8 & -.7 \\ .7 & .8 \end{bmatrix} L$
   (d) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} L + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} L^2$

(2) Find the fundamental moving average representation corresponding to this non-fundamental one:

$$y(t) = \varepsilon(t) + \begin{bmatrix} 1.1 & 0 \\ 0 & .8 \end{bmatrix} \varepsilon(t-1)$$

$$\operatorname{Var}(\varepsilon(t)) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$