ON THE GENERICITY OF THE WINDING NUMBER CRITERION FOR LINEAR RATIONAL EXPECTATIONS MODELS

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ABSTRACT. In a recent paper Onatski derives a new criterion for existence and uniqueness of solutions of rational expectations models. Specialized to finite order models, the criterion is an improvement on the usual root-counting criterion, but shares its main defect — there are models on which it gives the wrong answer. Onatski argues that the models where the winding number gives the right answer are "generic" — an open, dense subset of the space of all models. This could give a mistaken impression. A sequence of models for which the new criterion works that converges in Onatski's metric to a model on which the criterion does not work shows increasingly bizarre solution behavior as the limit is approached. In a metric that treats models with very different solution behavior as very far apart, the sequence is divergent, not convergent. Models on which the winding number gives the wrong answer will not in fact be extremely uncommon in economics, and they are not in any substantively meaningful sense close to nicely behaved models for which the winding number gives the right answer.

I. A BARE-BONES EXAMPLE

Here's a linear rational expectations model that has no stable solution for which the Blanchard and Kahn (1980) regularity conditions do not hold and that is not "generic" in the terminology of Onatski (2006).

$$x_t = 1.1x_{t-1} + \varepsilon_t \tag{1}$$

$$E_t y_{t+1} = .9y_t + \nu_t\,. \tag{2}$$

ε_t and ν_t are exogenous, non-explosive stochastic processes. We are looking for a solution that does not show exponential growth in any linear combination of variables. Obviously here we have two unrelated equations. In the first, there is an unstable root and no expectational term. No expectational effects can save this equation from being explosive, with solutions growing at the rate $(1.1)^t$. The second equation has no unstable roots, yet includes an expectational term. There is no way to import the unstable root from the other equation and make it produce

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determinacy in this one. Every solution for y can be modified by the addition to the right-hand side of (2) of an arbitrary process η_t satisfying $E_t\eta_{t+1} = 0$. The Blanchard-Kahn rule, that the number of unstable roots must match the number of forward-looking variables, is satisfied, yet there is no stable solution. The reason is that the unstable root occurs in a part of the system that is decoupled from the expectational equation.

It is true, as Onatski claims, that in a certain sense the system (1-2) lies "close" to systems in which there is no such decoupling problem with the usual method of matching unstable roots to counts of forward-looking variables. But perturbations of the *non-zero* coefficients in the system will not do the trick, nor will perturbations of the past or future zero coefficients on x in (1) or on y in (2) do the trick. After all, each of the two separate equations is a one-variable system that is, by itself, generic, and as Onatski points out, generic systems remain generic under small perturbations of their coefficients. What is required is a perturbation that introduces y into (1).

So what kind of a generic model lies close, in this sense, to our non-generic model? Consider this "neighbor" model:

$$x_t = 1.1x_{t-1} - .000001y_t + \varepsilon_t \tag{3}$$

$$E_t y_{t+1} = .9y_t + \nu_t\,. \tag{4}$$

Nothing has changed except for the addition of a tiny coefficient on current y in the x equation. This model is generic. It has the same pair of stable and unstable roots as the unperturbed model. It has a unique, stable solution. For the case of serially independent, mean-zero disturbances, here it is:

$$x_t = .90x_{t-1} + .82\varepsilon_t + \frac{.000001}{1.1}\nu_t \tag{5}$$

$$y_t = 200{,}000x_{t-1} + 181{,}828\varepsilon_t - \frac{1}{1.1}\nu_t\,. \tag{6}$$

If we made the coefficient on y_t smaller in (3), the coefficients on x_{t-1} and ε_t in (6) would be even bigger. With systems responding this powerfully to shocks, we are likely to be distrustful of the linearity assumption, and other aspects of the solution as well.

Even within sets of models with unique stable solutions, all bounded by the same constant, models can be arbitrarily close in the ℓ_1 metric on $A(L)$ without being close in solution space. Here's an example of that:

$$x_t = x_{t-1} + ay_{t-1} + \varepsilon_t \tag{7}$$

$$E_t y_{t+1} = y_t + ax_t + \nu_t\,. \tag{8}$$

$$\tag{9}$$

At $a = 0$, this system has a unit root and root-counting approaches give no answer, but two models of this form, one with $a > 0$ and the other with $a < 0$ but the same absolute value, approach each other in the elementwise ℓ_1 metric on $\{A_s\}$

as $|a| \to 0$, while the solutions for positive and negative a remain far apart. The solution approaches, as $a \downarrow 0$, a system in which $x_t - y_t$ has a unit root and $x_t + y_t$ is serially uncorrelated, while as $a \uparrow 0$, it approaches a system in which $x_t + y_t$ has a unit root and $x_t - y_t$ is serially uncorrelated.

Economic models often have large numbers of zeros or other constraints on parameters, and these constraints often have a foundation in the model's underlying theory. Models for which root-counting and the winding number criterion fail to deliver the correct answer will always involve some form of decoupling of the system, so that unstable roots in one part of the model cannot be matched up with forward-looking variables or equations in another part of the model. Such decoupling is not at all unlikely in economic models that are constrained by theory, and alterations of the model that undo this decoupling, even when they involve small deviations from the model's constraints on the parameters, are likely to affect drastically both the model solution and the model's economic interpretation.

Even in models with finitely many leads and lags, there are some cases where the winding number criterion gives the correct answer when Blanchard-Kahn root-counting does not. But neither can be safely relied upon in general. The models where they give the wrong answer can easily arise in economic research. This implies that a reliable check for existence and uniqueness requires not just looking at the winding number, but at all the "partial indices" of the system, to make sure none have opposite signs. It is also possible, for systems with finitely many leads and lags, to check existence and uniqueness reliably using the `gensys.m` or `gensys.R` programs at `sims.princeton.edu/yftp/gensys`. The cases where the winding number or root counting give the wrong answer are those where these programs return (0,0) as the `eu` vector, which can be read as a result that there is neither existence nor uniqueness. The `gensys` program separates the stable and unstable parts of the system and checks whether i) the expectational terms enter the unstable part in a rich enough way to stabilize it (existence) and ii) whether the expectational terms that are pinned down in the unstable part by the stability requirement determine the expectational terms that appear in the stable part of the system (uniqueness). For finite-order systems, this is equivalent to checking signs of partial indices.

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