ANSWERS FOR EXERCISES A AND B

Exercise A: Primary surplus is $\tau$, now and forever. Required expected real return is $\rho$. Primary surplus will support current debt forever with no default risk (i.e. with $R_t \equiv \rho$):

$$b_t = \bar{b} = R \bar{b} - \tau = \rho \bar{b} - \tau$$

Now suppose that bondbuyers become nervous, think there might be a default, so require an interest rate $R > \rho$ in order to hold debt. If $b_t = \bar{b}$, then $b_{t+1} = R \bar{b} - \tau > \rho \bar{b} - \tau = \bar{b}$.

Prove: So long as $b_t > \bar{b}$ and $R > \rho$, $b_{t+1}/b_t > 1$ and $b_{t+1}/b_t$ increases as $b_t$ increases.

This is a case with real debt, so there is no $P_t$ in the budget constraint, and it can be written as

$$b_t = R b_{t-1} - \tau.$$ 

We will divide the constraint by $b_{t-1}$, then use the fact that $R > \rho$, then use the fact that $b_{t-1} > \bar{b}$.

$$\frac{b_t}{b_{t-1}} = \frac{R - \tau}{b_{t-1}} > \frac{\rho - \tau}{\bar{b}} > \frac{\rho - \tau}{\bar{b}} = \frac{\rho - \tau}{\rho} = 1.$$ 

Furthermore, we can see from the first equality above that $b_t/b_{t-1}$ is increasing in $b_{t-1}$.

Exercise B: Suppose $\tau_0$ is 10, $\tau_1$ is 8, the probability of switching from $\tau_0$ to $\tau_1$ permanently is .1 every period, the amount of nominal debt initially is $B_0$, the real gross rate of return is $\rho$, and the nominal gross interest rate is $R$. Find the inflation or deflation every period before the drop, and the inflation rate at the date of the drop. Assume that after the drop everyone is certain that $\tau_1$ will prevail forever.

The budget constraint is

$$b_t = R \frac{P_{t-1}}{P_t} b_{t-1} - \tau_t.$$ 

Since there is a real investment alternative delivering a gross return $\rho$, we assume that investors insist that the expected yield on government debt is also $\rho$, so

$$E_{t-1} \left( \frac{RP_{t-1}}{P_t} \right) = \rho.$$ 

Then before the switch we have

$$E_{t-1} b_t = \rho b_{t-1} - E_{t-1} \tau_t = \rho b_{t-1} - .9 \tau_0 - .1 \tau_1 = \rho b_{t-1} - 9.8.$$
So long as $b_t$ can’t explode exponentially faster than $\rho^t$, we can solve this forward to get a before-the-switch value of real debt:

$$b_0 = \frac{9.8}{\rho - 1}.$$ 

After the switch, with $\tau$ fixed at $\tau_1 = 8$, solving forward gives us

$$b_1 = \frac{8}{\rho - 1}.$$ 

At the date of the switch, we will have

$$b_t = b_1 = R \frac{P_{t-1}}{P_t} b_0 - \tau_1.$$ 

Since we have already found formulas for $b_0$ and $b_1$, we can solve this for the value of $\Pi_1 = P_t / P_{t-1}$ at the time of the switch:

$$\Pi_1 = \frac{b_0 R}{b_1 + \tau_1} = \frac{\frac{9.8R}{\rho - 1}}{\frac{8}{\rho - 1} + 8} = \frac{9.8R}{8\rho}.$$ 

When the switch has not occurred, we will instead have

$$b_t = b_0 = R \frac{b_0}{\Pi_0} - \tau_0,$$

which we can solve to find

$$\Pi_0 = \frac{9.8}{10 - \frac{.2}{\rho}}.$$ 

When $\rho = R$, the gross inflation at the time of the switch is $9.8/8 = 1.225$, and before the switch is

$$\frac{9.8}{10 - \frac{.2}{\rho}} < 1,$$

so there is slight deflation at every date before the switch, then a substantial inflation at the date of the switch.