First Order Conditions for Stochastic Problems: Examples*

1. Permanent income model with no-Ponzi condition

A Ponzi scheme is a business plan that involves borrowing and repaying the loans by further borrowing rather than by profitably investing the borrowed funds. Of course such a scheme has to come to an end, at which point someone loses a lot of money. Rational lenders will not lend money to a Ponzi scheme, instead insisting that the borrower behave in such a way as to be able to at least eventually repay his loans. This condition is imposed in (4) below. It contrasts with the condition usually imposed in mechanical linear-quadratic formalizations of this problem, in which instead the requirement

$$\beta^{\frac{1}{2}t} W(t) \to 0 \tag{1}$$

is imposed. This is a more stringent limit on borrowing than the no-Ponzi condition, and it at the same time imposes a limit on upward growth in W that makes no economic sense as a constraint.

Objective function:

$$\max_{\{C(t),W(t)(t)\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t U\left(C\left(t\right)\right)\right] \tag{2}$$

Budget constraint:

$$W\left(t\right) = \left(1 + r(t-1)\right)\left(W(t-1) - C\left(t-1\right)\right) + Y\left(t\right)\ . \tag{3}$$

No-Ponzi Condition:

$$\lim_{t\to\infty} \prod_{s=0}^{t-1} (1 + r(s))^{-1} W(t) \geq 0\,. \tag{4}$$

FOC's:

$$\partial C: \qquad U'(C(t)) = \beta(1 + r(t))E_t[\lambda(t+1)] \tag{5}$$

$$\partial W: \qquad \lambda(t) = \beta(1 + r(t))E_t[\lambda(t+1)] \tag{6}$$

Transversality:

$$\limsup_{T\to\infty} \beta^T E[U(T)'(\hat{C}(T) - \bar{C}(T)) - U(T)' \cdot (\hat{W}(T) - \bar{W}(T))] \leq 0 \tag{7}$$

for any feasible $\hat{W}$ sequence different from the optimal sequence $\bar{W}$.

1.1. **Solution for the quadratic utility, $1+r \equiv \beta^{-1}$, i.i.d. $Y(t)$ case.** We assume here the particular quadratic utility function $U(C(t)) = C(t) - \frac{1}{2}C(t)^2$. In this case, as we have seen in class, the two FOC's (5) and (6) reduce to

$$C(t) = E_t C(t+1) . \tag{8}$$

The transversality condition (7) becomes

$$\limsup_{T\to\infty} \beta^T \{(1-\bar{C}(T)) \cdot (\hat{C}(T) - \bar{C}(T) - \hat{W}(T) + \bar{W}(T))\} \leq 0 . \tag{9}$$

The conventional solution to this problem, satisfying (1), is found, as we have discussed in class, by solving the budget constraint forward, using (8) and the fact that, because Y is assumed i.i.d., $E_t Y(t+s) = \bar{Y}$ for $s \geq 1$. The result is

$$C(t) = (1-\beta)W(t) + \beta\bar{Y} , \tag{10}$$

where $\bar{Y}$ is the constant $E[Y(t)]$. Remember that to derive this we *assumed* that $E[\beta^t W(t)]$ converged to zero. If we substitute (10) into the budget constraint (3), we get

$$W(t) = W(t-1) + Y(t) - \bar{Y} . \tag{11}$$

Since Y is i.i.d., the variance of $W(t)$ can be seen from (11) to grow linearly with t, so (1) is satisfied, and thus *a fortiori* the no-Ponzi condition (4) is satisfied.

So we have a candidate solution to our problem with the no-Ponzi constraint. It satisfies the Euler equation (8) by construction, and it satifies a conventional transversality condition requiring that W not explode exponentially.

There are two ways to see that this is not a solution to the problem as posed. One is to look at the correct transversality condition (9). Our objective function has a satiation point at $C(t) = 1$, beyond which the marginal utility of consumption becomes negative. It can be shown that for a random walk with i.i.d. increments (which is what (11) says W is) the probability of crossing any particular fixed value infinitely often is one. In other words, with probability one there will be infinitely many dates t at which we have $C(t) = (1-\beta)W(t) + \beta\bar{Y} > 1 + \varepsilon$, where $\varepsilon > 0$ is some arbitrary fixed number. At any given date t it is possible to decrease $C(t)$ to the satiation level 1 and simultaneously increase $W(t)$ by the same amount. This only makes $W(t)$ larger, and if future values of consumption are left unchanged, the effect on $W(t+s)$ for $s > 0$ is to make those values larger.[1] Thus deviating from our candidate optimal policy in this way will not violate the no-Ponzi condition, which only restricts downward explosion, not upward explosion, in W. But because the budget constraint is an unstable difference equation in W, even a single deviation of this sort sets off exponential growth in W at the rate β^{-1}, making the second term in the transversality condition (9) have a positive lim inf along the subsequence of t's at which $C(t) > 1 + \varepsilon$.

While this result may help us understand the limits of the usual transversality condition, it does not actually prove that the conventional non-explosive solution is not a

[1] Remember that we are constructing an alternative feasible path here, along which the budget constraint (3), but not the optimal policy rule (10) will be satisfied

solution to the model with the no-Ponzi constraint. That is because the Euler equations plus the no-Ponzi condition (plus convexity and concavity conditions) are only sufficient conditions, not necessary conditions, for an optimum. But we can finish the argument by comparing welfare on our deviant consumption path to welfare on the non-explosive solution to the Euler equations. Our deviant path involves only raising $U(C(t))$ by reducing $C(t)$ to 1 whenever it exceeds that value. The only consequence of this is to raise utility at some dates, while causing W to explode upward. The path is feasible, and must have strictly higher utility than the nonexplosive solution. Thus the non-explosive "solution" is not in fact an optimum.

So what would a true solution look like? Unfortunately, there is no analytic solution available, no matter what simplifying assumptions we make on the distribution of Y. We do know the following facts, however. The first order condition (5) implies that C will be a martingale. It is clear that on an optimal path $C(t) \leq 1$ at all times. (Otherwise our strategy of reducing C to 1 at all dates when it exceeds 1 would again be a feasible improvement on the path.) So C is a martingale bounded above by 1. The martingale convergence theorem, which we have already cited in class, asserts that bounded martingales converge with probability 1. So with probability 1 C converges to something. If it does so, it is not hard to show that W must blow up exponentially, and the only feasible direction for it to do this is upward, because of the no-Ponzi condition. But then if the limiting value of C is less than one, it is possible to improve on it by increasing $C(t)$ at some date, and because this would only somewhat lower an already upward-explosive path for W, it would be feasible. So the only limit C can converge to on an optimal path is 1. Note that if Y is bounded below by zero, say, then when $W(t)(1-\beta) \geq 1$, it is possible to set $C(s) = 1$, all $s \geq t$, without any risk that W will ever become negative. (The budget constraint (3) implies that under these conditions W is with probability one non-decreasing.) It is possible to use this fact to show that on every optimal path there is some date T^* after which (i.e. for $t > T^*$) $C(t) \equiv 1$.

Summarizing, the true solution to this problem makes $C(t)$ follow a martingale process that eventually reaches the satiation level 1 and sticks there. On this solution path, W eventually explodes upward at the rate β^{-t}. This is in contrast to the non-explosive solution to the Euler equations, along which C exceeds the satiation level infinitely often.

1.2. **Linearizing the general model.**

The linearized version of the Euler equations (5)-(6) and the constraint (3) reads

$$U''(\bar{C})dC(t) = \beta \cdot (1+\bar{r})E_t[d\lambda(t+1)] + \beta\bar{\lambda}dr(t) \tag{12}$$

$$d\lambda(t) = \beta \cdot (1+\bar{r})E_t[d\lambda(t+1)] + \beta\bar{\lambda}dr(t) \tag{13}$$

$$dW(t) = (1+\bar{r})(dW(t-1) - dC(t-1)) + (\bar{W} - \bar{C})dr(t-1) + dY(t)\,. \tag{14}$$

In these linearized equations, a term of the form $dZ(t)$ stands for $Z(t) - \bar{Z}$, i.e. simply the deviation between $Z(t)$ and its value at the point we are linearizing around.

Collecting these equations into matrix notation, we have

$$\begin{bmatrix} 0 & 0 & \beta(1+\bar{r}) \\ 0 & 0 & \beta(1+\bar{r}) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} dC(t) \\ dW(t) \\ d\lambda(t) \end{bmatrix} =$$

$$\begin{bmatrix} U''(\bar{C}) & 0 & 0 \\ 0 & 0 & 1 \\ -(1+\bar{r}) & 1+\bar{r} & 0 \end{bmatrix} \begin{bmatrix} dC(t-1) \\ dW(t-1) \\ d\lambda(t-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \eta(t) + \begin{bmatrix} 0 & -\beta\bar{\lambda} \\ 0 & -\beta\bar{\lambda} \\ 1 & \bar{W}-\bar{C} \end{bmatrix} \begin{bmatrix} dY(t) \\ dr(t-1) \end{bmatrix} . \quad (15)$$

Notice that, though there are two equations with dummy error terms, they both involve the same expectational error, so there is only dummy disturbance η. Notice also that the coefficient matrix on the left of the equality, which is playing the role of Γ_0 in the notes, is singular. To apply our Jordan-decomposition based methods, we have to modify the system to get rid of the singularity (though `gensys.m` and matlab could proceed with the system as it is). Usually the best way to do that is to eliminate variables that can be solved for analytically. Here we use the fact that (5) and (6) together imply $\lambda(t) = U'(C(t))$, so that $d\lambda(t) = U''(\bar{C})dC(t)$. This results, after elimination of λ, in the reduced system

$$\begin{bmatrix} U''(\bar{C})\beta(1+\bar{r}) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dC(t) \\ dW(t) \end{bmatrix} =$$

$$\begin{bmatrix} U''(\bar{C}) & 0 \\ -(1+\bar{r}) & (1+\bar{r}) \end{bmatrix} \begin{bmatrix} dC(t-1) \\ dW(t-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \eta(t) + \begin{bmatrix} -U'(\bar{C})\beta & 0 \\ \bar{W}-\bar{C} & 1 \end{bmatrix} \begin{bmatrix} dr(t-1) \\ dY(t) \end{bmatrix} . \quad (16)$$

Using Γ_0 to refer to the first square matrix on the left of the equation and Γ_1 for the first square matrix on the right, we can find

$$A = \Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} \beta^{-1}(1+\bar{r})^{-1} & 0 \\ -(1+\bar{r}) & (1+\bar{r}) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \dfrac{1}{1-\beta^{-1}(1+\bar{r})^{-2}} & 1 \end{bmatrix} \begin{bmatrix} \beta^{-1}(1+\bar{r})^{-1} & 0 \\ 0 & (1+\bar{r}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{-1}{1-\beta^{-1}(1+\bar{r})-2} & 1 \end{bmatrix} , \quad (17)$$

where the expression on the right is the Jordan decomposition of A. Because a triangular matrix has its eigenvalues on the diagonal, we can see immediately from A, before computing the Jordan decomposition, that it has one unstable root of $1+\bar{r}$, and, if $\beta \cdot (1+\bar{r}) > 1$, one stable root of $\beta^{-1}(1+\bar{r})^{-1}$. If U does not show satiation, then the unstable root is unlikely to be consistent with optimization. By the usual "root-counting" condition, then, we have one unstable root to correspond to our one dummy error term η, and should most likely have existence and uniqueness. In a model with a single unstable root like this, the criterion is that after multiplying the system by P^{-1}, the right-hand matrix in (17) (the matrix whose rows are the left-eigenvectors), there should be a non-zero coefficient on η in the unstable equation in the transformed system. It is easily verified that this condition is satisfied here.

What if $\beta^{-1} \cdot (1+\bar{r})^{-1} \geq 1+\bar{r}$, so that there are two unstable roots as large as the average interest rate? We can still "solve forward", but we are likely to run into existence problems. In this case the whole system is unstable, so the condition for existence is that the column of

$$P^{-1}\Pi = \begin{bmatrix} \dfrac{1}{\dfrac{-1}{1-\beta^{-1}(1+\bar{r})^{-}2}} & \begin{matrix} 0 \\ 1 \end{matrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\dfrac{-1}{1-\beta^{-1}(1+\bar{r})^{-}2}} \end{bmatrix} \tag{18}$$

should span a space including the columns of

$$P^{-1}\Psi = \begin{bmatrix} \dfrac{1}{\dfrac{-1}{1-\beta^{-1}(1+\bar{r})^{-}2}} & \begin{matrix} 0 \\ 1 \end{matrix} \end{bmatrix} . \tag{19}$$

But the square matrix in (19) is by construction non-singular, therefore spans the whole of $\mathbb{R}^2$, so the condition for existence cannot be met.

This situation of $\beta < (1+\bar{r})^{-2}$ implies very strong discounting of the future, relative to the rate of return on investment or the interest rate on loans. There is no optimal policy, because the nature of good policies is to consume a large amount right away by borrowing, then postpone repaying the loans a long time. For any consumption path of this type, there is always another one that consumes even more immediately and postpones the repayment even longer, and thereby improves on the original path. Postponing repayment raises utility even though the accumulating interest requires a greater future consumption sacrifice the longer payment is postponed. The discount factor β is so small that the increased future costs of repayment are offset by the increased discounting.

Problem 1 Redo the analysis above with the constraint (3) replaced by

$$H(t) = (1 + r(t-1))H(t-1) - C(t) + Y(t)\,, \tag{20}$$

and the no-Ponzi condition applied to H instead of W. This model is actually exactly the same as the one discussed above. W and H are just different ways of defining wealth, with both implying the same consumption possibility set. However the algebraic forms of the FOC's, transversality conditions, and Jordan decomposition are different.

2. Optimal Growth

The optimization problem is

$$\max_{\{K(t),I(t),C(t),L(t)\}} E\left[\sum_{t=0}^{\infty} \beta^t U\left(C\left(t\right), 1 - L\left(t\right)\right)\right] \tag{21}$$

subject to

$$C(t) + I(t) = A(t)f(K(t-1), L(t))\,, \tag{22}$$

$$K\left(t\right) = \left(1-\delta\right)K\left(t-1\right) + I(t) \tag{23}$$

and

$$K(t) \geq 0\,. \tag{24}$$

Problem 2: Find the Euler equations and the transversality condition for this problem.

Problem 3: Assume the marginal utility of leisure is zero, so that optimal L is identically 1 and it drops out of the decision problem in the growth model. Suppose $U(C(t)) = \log(C(t))$, $f(K(t-1), 1) = K(t-1)^{\alpha}$, and $\delta = 1$. Show that a constant $K(t)/C(t)$ satisfies the FOCs, including transversality, and that the convexity and concavity conditions are also satisfied, so that the constant K/C solution is in fact an optimum. Assume $0 < \alpha < 1$. Assume the exogenous stochastic process A is bounded away from zero and infinity.