Econ. 511b Spring 1999 George Hall and Chris Sims

Answers for Decentralization and Asset Pricing Exercise*

1. The consumer's first-order conditions deliver

$$\frac{Q_t}{C_1(t)} = \beta E_t \left[\frac{Q_{t+1} + Y_2(t+1)}{C_1(t+1)} \right] . \tag{A1}$$

The consumer's budget constraint is

$$C_1(t) = Y_1(t) + Y_2(t) , \tag{A2}$$

where we have used the fact that $S_1(t) = 1$ for all t. This lets us rewrite (A1) as

$$Q_t = \beta E_t \left[\frac{(Y_1(t) + Y_2(t)) \cdot (Q_{t+1} + Y_2(t+1))}{Y_1(t+1) + Y_2(t+1)} \right] \tag{A3}$$

and then solve it forward to obtain

$$\begin{aligned} Q_t &= E_t \left[(Y_1(t) + Y_2(t)) \sum_{s=1}^{\infty} \beta^s \frac{Y_2(t+s)}{Y_1(t+s) + Y_2(t+s)} \right] \\ &= (Y_1(t) + Y_2(t)) \frac{E\left[\frac{Y_2(t)}{Y_1(t)+Y_2(t)} \right]}{\beta^{-1} - 1} . \end{aligned} \tag{A4}$$

In deriving the last equality in (A4) we are using the fact that the Y's are all i.i.d., so the random ratios in the forward solution all have the same expectation conditional on information at t. So we've now answered the question.

2. The steady state of the model with Y's having mean 1 is $C_1 \equiv C_2 \equiv 1$, with $Q \equiv 1/(\beta^{-1} - 1) \equiv 9$. The four equations we will use are the social resource constraint (49) (in the notes), the individual's constraint (47) with $i = 1$, and the Euler equations with respect to S (54) for $i = 1, 2$, with the Lagrange multiplier substituted out using the C first order conditions (53). The resulting system, arranged to have most recently dated variables on the left (mostly), and with endogenous error terms inserted to allow us to drop the E_t's, is

$$C_1(t) + C_2(t) = Y_1(t) + Y_2(t) \tag{A5}$$

$$C_1(t) + Q_t S_1(t) - Y_1(t) = (Q_t + \delta_t) S_1(t-1) \tag{A6}$$

$$\beta \frac{Q_t + \delta_t}{C_1(t)} = Q_{t-1} C_1(t-1) + \eta_1(t) \tag{A7}$$

$$\beta \frac{Q_t + \delta_t}{C_2(t)} = Q_{t-1} C_2(t-1) + \eta_2(t) . \tag{A8}$$

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Linearizing these equations around the steady state produces a system in our canonical form

$$\Gamma_0 x_t = \Gamma_1 x_t - 1 + \Psi z_t + \Pi \eta_t \tag{A9}$$

if we define $x_t = [dC_1(t)\, dC_2(t)\, dQ_t\, dS_1(t)]'$ and $z_t = [dY_1(t)\, dY_2(t)\, d\delta_t]'$ and set

$$\Gamma_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 9 \\ 9 & 0 & -.9 & 0 \\ 0 & 9 & -.9 & 0 \end{bmatrix}, \qquad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 9 & 0 & -1 & 0 \\ 0 & 9 & -1 & 0 \end{bmatrix},$$
$$\Psi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & .9 \\ 0 & 0 & .9 \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A10}$$

It is possible, by taking the difference of the last two equations and re-ordering equations and variables, to get this model into a form in which Γ_0 is block triangular, with two 2×2 blocks on the diagonal. It then becomes possible to deal with the system analytically. However, we are going to proceed by brute force numerical methods. The analytic approach would have advantages if we wanted to make a more complete analysis of how the system's properties depend on choices of parameters.

The system matrix has the Jordan decomposition

$$\Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} 0.5000 & -0.5000 & 0 & 0 \\ -0.5000 & 0.5000 & 0 & 0 \\ -5.0000 & -5.0000 & 1.1111 & 0 \\ -0.0556 & 0.0556 & 0.0000 & 1.1111 \end{bmatrix} = V \Lambda V^{-1}, \tag{A11}$$

with the eigenvalues arranged along the diagonal of Λ as [1.1111, 1.1111, 1, 0] and

$$V = \begin{bmatrix} 0 & 0 & -0.5774 & 0.1098 \\ 0 & 0 & 0.5774 & 0.1098 \\ 0 & 0.9929 & 0 & 0.9879 \\ 1.0000 & -0.1193 & -0.5774 & -0.0000 \end{bmatrix}. \tag{A12}$$

We will also need to use the matrix whose rows are the left eigenvectors,

$$V^{-1} = \begin{bmatrix} -1.0407 & -0.0407 & 0.1202 & 1.0000 \\ -4.5324 & -4.5324 & 1.0072 & 0 \\ -0.8660 & 0.8660 & 0 & 0 \\ 4.5552 & 4.5552 & 0 & 0 \end{bmatrix} \tag{A13}$$

To find the full solution we have to "solve the unstable equations forward", meaning we make the change of variables $w_t = V^{-1} x_t$, so that the transformed

system becomes

$$w_t = \begin{bmatrix} 1.1111 & 0 & 0 & 0 \\ 0 & 1.1111 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w_{t-1} + V^{-1}\Psi dz_t + V^{-1}\Pi\eta_t \,. \tag{A14}$$

Because our linearized system has i.i.d. z's as well as serially uncorrelated η's, the forward solution for the two unstable components of w, w_1 and w_2, has no stochastic component (the expected future values are all identically equal to unconditional means) and simply sets w_1 and w_2 identically equal to their steady state values. In other words,

$$\begin{bmatrix} dw_{1t} \\ dw_2(t) \end{bmatrix} = \begin{bmatrix} -1.0407 & -0.0407 & 0.1202 & 1.0000 \\ -4.5324 & -4.5324 & 1.0072 & 0 \end{bmatrix} dx_t = 0; \tag{A15}$$

Combining this pair of equations (A15) with two of the original equations gives the answer. (I will supply a more detailed completion of the answer by next Tuesday. This is as far as I can get today under a time constraint.) If one wants to check to be sure that there is existence and uniqueness here, one checks whether the 2×2 matrix in the first two rows of $V^{-1}\Pi$ (in (A14)) is full rank, which it is.