1. (a) An Arrow-Debreu competitive equilibrium is a price sequence $\{p_t^0\}_{t=0}^{\infty}$ and an allocation $\{c_t^o, c_t^e\}_{t=0}^{\infty}$ such that:

   i. given the price sequence, the allocation solves the optimum problem for each agent:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t^i$$

   subject to

$$\sum_{t=0}^{\infty} p_t^0 c_t^i \leq \sum_{t=0}^{\infty} p_t^0 y_t^i.$$

   ii. the allocation is feasible:

$$c_t^o + c_t^e = y_t^o + y_t^e \quad \forall \;\; t.$$

   In an Arrow-Debreu economy all trades take place at time 0 prior to any date 0 consumption. During this fictional time 0 marketplace, agents buy and sell claims to future consumption for all dates $t = 0, 1, ..., \infty$ at prices p_t^0. If $\alpha > \gamma$ the odd agents sell claims to the consumption good for all future even periods to the even agents. The odd agents then purchase claims to the consumption good for all future odd periods from the even agents. If $\alpha < \gamma$, the situation is reversed.

   (b) Type i's Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln c_t^i + \lambda^i \sum_{t=0}^{\infty} p_t^0 (y_t^i - c_t^i).$$

   A first-order necessary condition for an optimum is:

$$\frac{\partial \mathcal{L}}{\partial c_t^i} \;\; = \;\; \beta^t \frac{1}{c_t^i} - \lambda^i p_t^0 = 0.$$

   This implies

$$c_t^i = \frac{\beta^t}{\lambda^i p_t^0}.$$

   This must hold for all i and t. So we know it holds for the odd agent:

$$\lambda^o c_t^o = \frac{\beta^t}{p_t^0}.$$

   Using the feasibility condition from our definition of an equilibrium we get:

$$\lambda^e (\alpha + \gamma - c_t^o) = \frac{\beta^t}{p_t^0}.$$

   So

$$\lambda^o c_t^o = \lambda^e (\alpha + \gamma - c_t^o),$$

   and

$$c_t^o = \frac{\lambda^e (\alpha + \gamma)}{\lambda^o + \lambda^e}.$$

No terms on the right hand side of this expression depend on t. So

$$c_t^o = c^o \quad \forall \ t,$$

and

$$c_t^e = \alpha + \gamma - c^o \quad \forall \ t.$$

Since the allocation is constant across time, from the FOC we get

$$\beta^t = kp_t^0$$

where k is a constant. We are free to normalize $kp_0^0 = 1$. So $p_t^o = \beta^t$.
For the odd agents, the present value of their endowment stream is:

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t^0 y_t^o &= \alpha + \beta\gamma + \beta^2\alpha + \beta^3\gamma + ... \\
&= \alpha + \beta^2\alpha + ... + \beta\gamma + \beta^3\gamma + ... \\
&= \frac{\alpha}{1-\beta^2} + \frac{\beta\gamma}{1-\beta^2} \\
&= \frac{\alpha + \beta\gamma}{1-\beta^2}.
\end{aligned}$$

Likewise for the even agents we get

$$\sum_{t=0}^{\infty} p_t^0 y_t^e = \frac{\gamma + \beta\alpha}{1-\beta^2}.$$

For the odd agents, the present value of the consumption stream is:

$$\sum_{t=0}^{\infty} p_t^0 c^o = \frac{c^o}{1-\beta}$$

From the odd agent's budget constraint we get

$$\frac{c^o}{1-\beta} = \frac{\alpha + \beta\gamma}{1-\beta^2}.$$

So

$$c^o = \frac{\alpha + \beta\gamma}{1+\beta}.$$

Following similar steps for the even agent yields

$$c^e = \frac{\gamma + \beta\alpha}{1+\beta}.$$

2. This question was answered well by most students, though most did not get a clean answer to part (c). The Euler equations are

$$\begin{aligned} \partial C:& \quad \tfrac{1}{C_t} = && \lambda_t && (1)\\ \partial B:& \quad \lambda_t = && \beta\cdot(1+r_t)E_t[\lambda_{t+1}] && (2)\\ \partial A_1:& \quad \lambda_t = && \beta\cdot E_t[\lambda_{t+1}Q_{t+1}] && (3)\\ \partial A_2:& \quad Q_t\lambda_t = && \beta\cdot(1+y_{t-1})^2E_t[\lambda_{t+1}]\,. && (4)\end{aligned}$$

From (4) and (2) (taking their ratios) we can conclude that

$$Q_t = \frac{(1+y_{t-1})^2}{(1+r_t)}\,. \tag{5}$$

This completes the answer to parts (a) and (b). For part (c) one can substitute (5) into (3) to arrive at

$$1+y_t = \sqrt{\frac{1}{\beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\cdot\frac{1}{1+r_{t+1}}\right]}}\,. \tag{6}$$

Since λ_{t+1}/λ_t is a stochastic discount factor, this expression has the requested form.

3. As was explained during the exam, the original problem setup used notation differently from what we have usually done in class, though it was not logically inconsistent. Our convention has been that exogenous variables dated t (like A_t in this model) are known at t, and that choice variables dated t are chosen using data on exogenous variables known at t, but cannot depend on exogenous variables dated later. If this convention is applied to the original problem statement, it implies that R_t is chosen with knowledge of A_t, even though the constraint makes clear that, since it is a function of C_t and R_t, R_{t+1} is known at t. That is, the specification implies A_t is known "one period in advance". This is a perfectly legitimate interpretation of the model, since no assumptions about what information is known when was stated explicitly. Everything you were asked to prove remains true under this interpretation. One could also have interpreted the dating the way it often is in the literature, so that R_{t+1} is thought of as "chosen at t", i.e. with knowledge of A_t but not A_{t+1}. This interpretation requires that you recognized that in this case the FOC w.r.t. $R(t)$ needs an "E_{t-1}" in front of it rather than an "E_t". It was to reduce this possible source of confusion that I said during the exam that the question was probably better interpreted as having a constraint $C_t \le A_t(R_{t-1}-R_t)^\alpha$. Some people appeared to have been helped by this announcement, but others who had begun correct analyses based on a legitimate interpretation of the question apparently started over after the announcement. Answers on the whole were quite good.

For the problem with the constraint announced in class, the FOC's are

$$\begin{aligned} \partial C:& \quad \tfrac{1}{C_t} && = \lambda_t && (7)\\ \partial R:& \quad \alpha\lambda_t A_t(R_{t-1}-R_t)^{\alpha-1} && = \beta\alpha E_t[\lambda_{t+1}A_{t+1}(R_t-R_{t+1})^{\alpha-1}+\mu_t\,, && (8)\end{aligned}$$

where μ is the multiplier on the $R_t \geq 0$ constraint and λ that on the production technology constraint. Everyone seemed to recognize that $0 \leq \alpha < 1$ should be assumed, to keep the technology concave, and that either the economics of the model interpretation or the mathematics implicitly requires $R_{t+1} < R_t$, as otherwise the technology constraint becomes uninterpretable (involving negative numbers to a fractional power). It is clear that the $R \geq 0$ constraint can never be binding, so $\mu \equiv 0$. The reason is that marginal utility is unbounded as $C \to 0$, so it is always possible to increase utility by delaying the period in which R reaches zero. Most people just asserted $\mu \equiv 0$ or assumed an internal solution and omitted μ from the FOC's, which was OK.

The TVC for this problem is

$$\limsup_{t\to\infty} \left\{ -\beta^t E_t \left[\lambda_t A_t \alpha (R_{t-1} - R_t)^{\alpha-1} dR_t \right] \right\} \leq 0 \,. \tag{9}$$

(The dC_t component vanishes, once we impose the ∂C Euler equation.) Because of the $R \geq 0$ constraint and the fact that marginal utility and marginal product are here by construction always positive, the "usual simplification" is posible, leading to

$$\lim_{t\to\infty} \left\{ \beta^t E_t \left[\lambda_t A_t \alpha (R_{t-1} - R_t)^{\alpha-1} R_t \right] \right\} = 0 \tag{10}$$

From (7) and the constraint we can conclude that

$$\lambda_t = \frac{1}{A_t (R_{t-1} - R_t)^\alpha} \,. \tag{11}$$

Using this expression in the TVC to substitute out λ, or else just replacing λ with $1/C$, we can see that the TVC requires that C or the rate of resource exhaustion not go to zero much faster than R, and in particular that it not go to zero while R remains asymptotically bounded away from zero. This makes sense: it is pointless to maintain indefinitely an unused stock of the resource. Two common mistakes in interpreting the TVC were to claim that it rules out unbounded accumulation of R (in this problem R can only go down, never up) and to claim that it is needed to make sure R remains non-negative (this is taken care of by the $R \geq 0$ constraint and is a condition for feasibility, not an implication of optimality).

Substituting out λ from (8) using (11) gives us

$$\frac{1}{R_{t-1} - R_t} = E_t \left[\frac{\beta}{R_t - R_{t+1}} \right] \,. \tag{12}$$

Multiplying this equation by β^t gives us the martingale conclusion asked for in (b).

Part (c) asked that you verify that $R_t = \beta^t R_0$ is the solution. The first step is just to check that the martingale condition (12) that summarizes the Euler equations is satisfied. It is, trivially, because $\beta^t (R_{t-1} - R_t)^{-1}$ is constant along the proposed path. The TVC is also easily verified. And since this is a problem with concave objective function and convex constraints, this is sufficient to guarantee that we have the solution. Note that this means that R follows a deterministic path. Consumption follows a path that is stochastic, varying with A_t.

The interpretation of the model that uses the original dating in the constraint but maintains the assumption that R_t (with the original dating) is chosen without knowledge of A_t is the same model as discussed above, just with different t subscripts. The interpretation that uses the original dating and assumes that A_t is known when R_t is chosen leads to the same FOC's and TVC's, except that there are no E_t operators needed in the ∂R FOC. This may look like it's going to change the solution, but with the log-utility, Cobb-Douglas technology forms assumed here, as we have already determined, the R time path is deterministic even if A_t is not known in advance. So the solution is exactly the same under either assumption about when A_t is known.

4. The Bellman equation is:

$$v(w) = \max_{\text{accept, reject}} \left\{ w + \beta\lambda(w) \int v(w')F^n(dw') + \beta(1-\lambda(w))v(w), c + \beta \int v(w')F^n(dw') \right\}$$

where w is the best offer in hand.

There was a small typo is this question. The last sentence of the first paragraph should have read " ... Prob$(w_i \leq w) = F(w)$."