# Comp Question 3 Answer, Followed by the Question

**The Answer:** a. In a model like this $L$, and thus $C$ is naturally bounded. Also, $B/P$ is bounded below because there will have to be a lower bound on net worth, or at least a no-Ponzi condition, to prevent individuals from borrowing without ever repaying. If there is a no-Ponzi condition rather than a lower bound on net worth, slowly exploding $B/P < 0$ would be allowed. $B/P$ exploding upward will be ruled out by transversality conditions. Individuals cannot find it optimal to accumulate arbitrarily large amounts of real wealth while their non-capital income, consumption and taxes remain bounded. To do so would "waste" some of their wealth.

b. The three roots of the system appear on the diagonal of $\Lambda$. There is one endogenous disturbance $\eta$ in the model, so to determine it we need one unstable root. So long at $\bar{R} > 1$ (which is given), it produces one unstable root. The unit root is present regardless of the model's parameters. To avoid having two unstable roots, and thus non-existence, we need therefore that the third root, $(1 + \gamma_1)/(1 + \gamma_0)$, be less than one, i.e. $\gamma_1 < \gamma_0$. The left eigenvector corresponding to the unstable root, the first row of $P^{-1}$, is a linear combination of $b - p$ and $c$, so if $w_{1t}$ has any unstable component, it will imply that either $c$ or real debt explodes at the rate $\bar{R}^t$, which was declared impossible in the problem statement.

In this problem with a single unstable root, the condition for existence is simply that the coefficient on $\eta$ in the $w_1$ equation be non-zero, and this also guarantees uniqueness.

Many people seemed to think the root of $\bar{R}$ did not need to be suppressed, despite the declared impossibility of growth at the rate $\bar{R}^t$. I realized on rereading my notes on linear RE systems that a literal reading of their summary results might have suggested this is true, so not much credit was deducted for misclassifying $\bar{R}$ as "stable". (A repaired version of the notes is on the web page now.)

c. The system involves the endogenous error term $\eta_t$. Given any solution path for the system, if we have no restrictions on rates of growth, we can generate another solution path by adding arbitrary i.i.d. variation to the path of $\eta$ and we will have a new solution to the equation system.

d. First we find the stable solution for $w_{1t}$, the unstable component of the model. Solving forward, we arrive at

$$w_{1t} = -\bar{R}^{-1}(\phi \cdot (\bar{R} - 1) + 1)r_t \, . \tag{A1}$$

Since the forward-solved equation has to hold both with and without the $E_t$ operator applied to it, $\eta_{t+1}$ must exactly cancel all the time-$t$ i.i.d. shocks on the right of the forward-solved equation. That is,

$$\eta_t = \frac{1-\bar{R}}{\mu\phi} s_t + \bar{R}^{-1} r_t - \tilde{a}_t \; . \tag{A2}$$

We can use (A2) to eliminate $\eta$ from the second and third equations of (7) (on the exam), after which (A1), together with these revised second two equations of (7), form a non-explosive stochastic difference equation system that characterizes the relation of the exogenous disturbances $z_t$ and $r_t$ to the endogenous variables $b$, $p$, and $c$, as requested. To convert this to a solution for the original variables in the $y$ vector, use $y_t = P w_t$.

Many people had the basic idea of solving the equation for the single unstable component of $w$ forward and using the stable part in its original form. This by itself earned substantial credit. Very few applied this idea in any detail to the specific problem at hand.

A common poor answer was one that suggested solving the entire $3 \times 3$ system forward, not recognizing that the presence of a stable root will make the forward solution fail to converge.

**The Question:** The following equation system can be deduced from the first-order conditions, market-clearing conditions, and constraints of a model with government debt, no money, no capital, log utility, and competitive firms that set the real wage equal to the marginal product of labor. Nominal wage adjustment is sticky, however, which leads to the equation labeled the Phillips Curve below. The $E_t$ operator conditions on all variables dated $t$ and earlier.

$$\text{GBC:} \qquad \frac{B_t}{P_t} + \tau_t = R_{t-1} \frac{B_{t-1}}{P_t} \tag{1}$$

$$\text{Saving:} \qquad \frac{1}{P_t C_t} = \beta R_t E_t \left[ \frac{1}{P_{t+1} C_{t+1}} \right] \tag{2}$$

$$\text{Phillips Curve:} \qquad \frac{P_t}{P_{t-1}} = \kappa \left( \frac{C_t}{A_t} \right)^{\gamma_0} \left( \frac{C_{t-1}}{A_{t-1}} \right)^{-\gamma_1} \; . \tag{3}$$

Government policy is assumed to set

$$R_t = \bar{R} e^{r_t} \tag{4}$$

$$\tau_t = \bar{\tau} e^{s_t} \; , \tag{5}$$

where $r_t$ and $s_t$ are i.i.d., zero-mean random variables. We assume $\bar{R} = \beta^{-1}$ and $\gamma_0 > 0$, $\gamma_1 > 0$, $\beta \in (0,1)$. This system of equations can be log-linearized about its deterministic steady state to become

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi z_t + \Pi \eta_t \; , \tag{6}$$

where

$$y_t = \begin{bmatrix} b_t \\ p_t \\ c_t \end{bmatrix} \qquad z_t = \begin{bmatrix} s_t \\ r_{t-1} \\ \tilde{a}_t \end{bmatrix}$$

$$\eta_t = \log(P_t C_t) - E_{t-1}[\log(P_t C_t)]$$

$$\Gamma_0 = \begin{bmatrix} 1 & \bar{R}-1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -\gamma_0 \end{bmatrix} \qquad \Gamma_1 = \begin{bmatrix} \bar{R} & 0 & \bar{R} \\ 0 & 1 & 1 \\ 0 & 1 & -\gamma_1 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 1-\bar{R} & \bar{R} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Pi = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} .$$

Here lower case $b$, $p$, and $c$ are deviations from determistic steady state of logs of the corresponding upper-case variables in (1)-(3), $r$ and $s$ are as defined in (4)and (5), and $\tilde{a}_t = -\gamma_0 \log(A_t) + \gamma_1 \log(A_{t-1})$. We assume $\tilde{a}$ to be i.i.d. with mean zero.

Using the fact that $\Gamma_0^{-1}\Gamma_1 = P\Lambda P^{-1}$, with

$$P = \begin{bmatrix} 1 & 1 & -\theta \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & -1 & \theta-1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \bar{R} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1+\gamma_1}{1+\gamma_0} \end{bmatrix} ,$$

the linearized system can be rewritten in terms of a new variable vector $w_t = P^{-1} y_t$ as

$$w_t = \Lambda w_{t-1} + \Psi^* z_t + \Pi^* \eta_t \, , \tag{7}$$

where

$$\Psi^* = \begin{bmatrix} 1-\bar{R} & \mu\phi & -\mu\phi \\ 0 & 1 & 0 \\ 0 & \phi & -\phi \end{bmatrix} \qquad \Pi^* = \begin{bmatrix} \mu\phi \\ 1 \\ \phi \end{bmatrix} . \tag{8}$$

In the expressions above $\theta = (\bar{R}-1)(1+\gamma_1)/\big(\bar{R}(1+\gamma_0) - 1 - \gamma_1\big)$ , $\phi = 1/(1+\gamma_0)$, and $\mu = \theta + \bar{R} - 1$.

a. In solving the linearized system we will assume that solutions that imply increase of $c$ at any exponential rate or $b - p$ at a rate $\bar{R}^t$ or greater cannot correspond to equilibria of the original model. What economic arguments justify such an assumption for this model?

b. Under what conditions on the model's parameters ($\bar{R} = \beta^{-1}$, $\gamma_0$, and $\gamma_1$) is there a unique non-explosive solution to the system of equations?

c. Explain why, when the conditions you gave in b are met, the equation system (7) (or (6)) does not have a unique solution in this model if we allow arbitrary exponentially exploding solutions.

d. Use (7) to find $w_t$ as a function of $w_{t-1}$, $s_t$, $r_t$, $r_{t-1}$, and $\tilde{a}_t$. Point out how this delivers a solution for $y_t$. You do not need to complete all the substitutions if you display them all. For example, you might display a solution for $w_t$ that still involves $\eta_t$ but have a separate equation showing $\eta_t$ as a function of the exogenous disturbances $s_t$, $r_t$, $r_{t-1}$, and $\tilde{a}_t$. [Note that $z_t$ contains $r_{t-1}$, which is known at date $t-1$, so that though $E_t z_{t+v} = 0$ for $v > 1$, this is not true for $v = 1$.]