Answers to ISLM(?) Model Exercise

1. In canonical form, the system reads

$$\begin{bmatrix} \alpha & \phi \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} y_{t+1} \\ \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\theta & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \phi \\ 0 \end{bmatrix} [r_t] + \begin{bmatrix} \eta_{t+1} \\ \nu_{t+1} \end{bmatrix} \tag{A1}$$

where the expectational errors are given by $\eta_{t+1} = \alpha\left[y_{t+1} - E_t y_{t+1}\right] + \phi\left[\pi_{t+1} - E_t \pi_{t+1}\right]$.

2. The matrix $A = \Gamma_0^{-1}\Gamma_1$ is given by

$$A = \begin{bmatrix} \alpha & \phi \\ 0 & \gamma \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -\theta & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha} & -\frac{\phi}{\alpha\gamma} \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\theta & 1 \end{bmatrix} = \begin{bmatrix} \frac{\gamma+\phi\theta}{\alpha\gamma} & -\frac{\phi}{\alpha\gamma} \\ -\frac{1}{\gamma}\theta & \frac{1}{\gamma} \end{bmatrix} \tag{A2}$$

The eigenvalues are defined by:

$$1 - \lambda(\gamma + \phi\theta + \alpha) + \lambda^2\alpha\gamma = 0$$

which yields

$$\lambda_{1,2} = \frac{1}{2\alpha\gamma}\left(\gamma + \phi\theta + \alpha \pm \sqrt{(\gamma^2 + 2\phi\theta\gamma - 2\alpha\gamma + \phi^2\theta^2 + 2\phi\theta\alpha + \alpha^2)}\right) \tag{A3}$$

If $\gamma = \alpha = 1$, then

$$\lambda_{1,2} = 1 + \frac{\phi\theta}{2} \pm \sqrt{\phi\theta + \frac{\phi^2\theta^2}{4}} \tag{A4}$$

Obviously, with $\phi > 0$ and $\theta > 0$, one root will always be larger than one, while the other is smaller than one.

3. If we let $q_t = p_{t-1}$ and solve the money demand system for r as

$$r_t = \frac{y_t + p_t - m_t}{\xi}\ , \tag{A5}$$

we can obtain the new system, in y, p, and q, with m exogenous,

$$y_t = \alpha y_{t-1} - \frac{\phi}{\xi}(y_t + p_t - m_t) + \phi p_{t+1} - \phi p_t + \eta_1(t+1) \tag{A6}$$

$$p_t = q_t + \gamma p_t + 1 - \gamma p_t + \theta y_t + \eta_2(t+1) \tag{A7}$$

$$q_t = p_{t-1}\ . \tag{A8}$$

Rewriting this in matrix notation, it is

$$\begin{bmatrix} \alpha & \phi & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t+1} \\ p_{t+1} \\ q_{t+1} \end{bmatrix} = \begin{bmatrix} 1+\dfrac{\phi}{\xi} & \dfrac{\phi}{\xi}+\phi & 0 \\ -\theta & 1+\gamma & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ p_t \\ q_t \end{bmatrix} + \begin{bmatrix} -\dfrac{\phi}{\xi} \\ 0 \\ 0 \end{bmatrix} m_t + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \eta(t)\,. \tag{A9}$$

It is easily seen that this is exactly in the standard form of equation (3) in the problem statement.

Forming $A = \Gamma_0^{-1}\Gamma_1$ for the particular numerical values given ($\gamma = \alpha = 1$ and $\xi = \theta = \phi = .5$), we find

$$A = \begin{bmatrix} 2.2500 & 0.5000 & -0.5000 \\ -0.5000 & 2.0000 & 1.0000 \\ 0 & 1.0000 & 0 \end{bmatrix}. \tag{A10}$$

According to Matlab, this matrix has eigenvalues 2.3086 + 0.3405i, 2.3086 - 0.3405i, and -0.3673. Two of these are much larger than one in absolute value, so the commonly cited condition for a solution, that the number of unstable roots match the number of η's, is satisfied here. Though you were not asked to do this, you could have used the results from our Tuesday, 1/19 lecture to check existence and uniqueness. The matrix with columns equal to the eigenvectors of A (also found by Matlab) is

$$V = \begin{bmatrix} 0.4074 - 0.4546i & 0.4074 + 0.4546i & 0.2381 \\ 0.6549 + 0.3180i & 0.6549 - 0.3180i & -0.3348 \\ 0.2975 + 0.0939i & 0.2975 - 0.0939i & 0.9117 \end{bmatrix} \tag{A11}$$

Multiplying the system through by V^{-1} gives us two equations with unstable roots above one with a stable root. The coefficient matrix on η in the first two equations is

$$V^{1\cdot} = \begin{bmatrix} 0.3654 + 0.7922i & 0.4967 - 0.3417i & 0.0870 - 0.3324i \\ 0.3654 - 0.7922i & 0.4967 + 0.3417i & 0.0870 + 0.3324i \end{bmatrix}. \tag{A12}$$

When this is multiplied times Π, the matrix of coefficients of η in the original system (A9), the result is just the first two columns of the matrix on the right of (A12), and it is easily seen that this is a square, non-singular matrix. This is sufficient to guarantee uniqueness and existence of a solution. (The more complicated column and row spanning conditions come into play only when this matrix is non-square or singular.)

It is worth noting that this result, that the model has a unique solution, is not generic. If instead we made $\alpha = 1.2$ (e.g. from a strong investment accelerator effect) and $\phi = .1$ (e.g. because investment and savings are interest-inelastic), then only one of the roots is unstable, so that the model contains an indeterminacy.