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1. Problem Set 9: Exercise on FTPL

Same model as in paper and lecture, only that one-period govenment bonds are replaced by consols, which are bonds that pay one dollar forever. It has current market value $1/r_t$, where r_t is an infinitely long interest rate.

The representative agent maximizes with respect to C, B, and M the objective function

$$E\left[\sum_{t=0}^{\infty} \beta^t \log\left(C_t\right)\right] \tag{1}$$

subject to the constraints, for $t = 0, ..., \infty$,

$$C_t\left(1 + \frac{\gamma v_t}{1 + v_t}\right) + \frac{B_t - B_{t-1}}{P_t r_t} + \frac{M_t - M_{t-1}}{P_t} = Y_t + \frac{B_{t-1}}{P_t} - \tau_t \tag{2}$$

$$B_t \geq 0 \tag{3}$$

$$v_t = \frac{P_t C_t}{M_t} \tag{4}$$

1.1. **Question 1.** Find the FOC's for an optimum in the agent's problem.

1.2. **Answer.** The FOC's are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t\left(1 + \frac{\gamma v_t}{1 + v_t}\right) + \pi_t \frac{P_t}{M_t} \tag{5}$$

$$\partial B: \qquad \frac{\lambda_t}{P_t r_t} = \beta E_t\left[\frac{\lambda_{t+1}(1 + r_{t+1})}{P_{t+1} r_{t+1}}\right] + \mu_t \tag{6}$$

$$\partial M: \qquad \frac{\lambda_t}{P_t} = \beta E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right] + \pi_t \frac{P_t C_t}{M_t^2} + \theta_t \tag{7}$$

$$\partial v: \qquad \pi_t = \lambda_t C_t \frac{\gamma}{(1 + v_t)^2} \tag{8}$$

Substituting π_t in the C and M-FOC's, and defining z_t as

$$z_t \equiv \frac{1}{v_t\left(1 + \frac{\gamma v_t}{1 + v_t} + \frac{\gamma v_t}{(1 + v_t)^2}\right)} \tag{9}$$

yields

$$\frac{z_t}{M_t}\left(1 - \gamma \frac{v_t^2}{(1 + v_t)^2}\right) = \beta E_t\left[\frac{z_{t+1}}{M_{t+1}}\right] \tag{10}$$

Now set $r_t \equiv \overline{r}$, this simplifies the B-FOC. Dividing the M-FOC by the B-FOC and assuming that both assets are strictly positive, so that the multipliers on the nonnegativity constraints are zero, yields.

$$1-\gamma\frac{v_t^2}{(1+v_t)^2}=\frac{1}{1+\overline{r}} \tag{11}$$

This ties down $v_t=\overline{v}$. Also, it follows that $z_t=\overline{z}$.

1.3. **Question 2.** Verify that, when initial $B>0$, the conditions under which there is a unique equilibrium price level under the policy combination $r_t\equiv\overline{r}$, $\tau_t\equiv\overline{\tau}$ are the same as those under which there is a unique equilibrium price level under $R_t\equiv\overline{R}$, $\tau_t\equiv\overline{\tau}$ in the one-period bond model of the lectures and the "Simple Model" paper.

1.4. **Answer.** The remaining FOC is

$$\left(1-\gamma\frac{\overline{v}^2}{(1+\overline{v})^2}\right)=\beta E_t\left[\frac{M_t}{M_{t+1}}\right] \tag{12}$$

With the money demand curve above, this is equivalent to

$$\frac{1}{(1+\overline{r})\beta}=E_t\left[\frac{M_t}{M_{t+1}}\right] \tag{13}$$

Note that with one-period bonds we would arrive instead at

$$\frac{1}{\beta\overline{R}}=E_t\left[\frac{M_t}{M_{t+1}}\right], \tag{14}$$

in which the one-period gross interest rate $\overline{R}$ simply plays the role of $1+\overline{r}$.

Equation (13) might seem consistent with a sunspot equilibrium, because as long as the expectation is satisfied, there is no violation of the FOC. However, consider the GBC

$$\frac{B_t-B_{t-1}}{P_t r_t}+\frac{M_t-M_{t-1}}{P_t}=\frac{B_{t-1}}{P_t}-\overline{\tau} \tag{15}$$

Multiply with P_t/M_t and take expectations E_{t-1} to get

$$E_{t-1}\left[\frac{B_t}{M_t\overline{r}}\right]-\frac{B_{t-1}}{\overline{r}}E_{t-1}\left[\frac{1}{M_t}\right]+1-M_{t-1}E_{t-1}\left[\frac{1}{M_t}\right]=B_{t-1}E_{t-1}\left[\frac{1}{M_t}\right]-\overline{\tau}E_{t-1}\left[\frac{P_t}{M_t}\right]$$

Rearranging, using the above conditions

$$E_{t-1}\left[\frac{B_t}{M_t\overline{r}}\right]=\frac{B_{t-1}}{M_{t-1}\overline{r}}\frac{1}{\beta}-1+\frac{1}{(1+\overline{r})\beta}-\overline{\tau v}\left(1+\frac{\gamma\overline{v}}{1+\overline{v}}\right)E_{t-1}\left[\frac{1}{Y_t}\right] \tag{16}$$

This is an unstable difference equation unless $B_t/(r_tM_t)$ is constant at its steady state value

$$\frac{B}{M\overline{r}}=\frac{1-\dfrac{1}{(1+\overline{r})\beta}+\overline{\tau v}\left(1+\dfrac{\gamma\overline{v}}{1+\overline{v}}\right)E_{t-1}\left[\dfrac{1}{Y_t}\right]}{\beta^{-1}-1}. \tag{17}$$

It is not hard to verify that, since $\overline{v}$ is increasing in $\overline{r}$ (from (11)), the right-hand side of this expression is increasing in $\overline{r}$. This reflects the fact that higher nominal interest rates increase the opportunity cost of holding money and correspond to higher steady-state inflation, which increases seignorage revenue and thereby backs higher levels of real debt.

If we make an analogous sequence of substitutions and expectation-takings for the model with one-period bonds, we arrive at, instead of (16),

$$E_{t-1}\left[\frac{B_t}{M_t}\right] = \frac{B_{t-1}}{M_{t-1}}\frac{1}{\beta} - 1 + \frac{1}{\overline{R}\beta} - \overline{\tau}\overline{v}\left(1+\frac{\gamma\overline{v}}{1+\overline{v}}\right)E_{t-1}\left[\frac{1}{Y_t}\right], \tag{18}$$

where in the one-period model B_t is the number of one-period bonds issued at t. In (18) we have R playing the role that $(1+r)$ plays in (16), and B_t, the number of one-period bonds (which for one-period bonds is of course the same thing as their market value) held by agents at t, playing the role that B_t/r_t, the nominal market value of consols held by agents at t, plays in (16). Thus the unique constant equilibrium value of $B/(Mr)$ that we display in (17) for the consol model is also the unique constant equilibrium value of B/M in the one-period bond model, so long as we keep $\overline{R} = 1+\overline{r}$.

Now there are two aspects of this result to check: First of all, is this solution for $B/(Mr)$ unique? If yes, what is the implied behavior of the price level?

Suppose B_t/M_t increases without bound, i.e., starts above its steady state value. From the definition of velocity, we have that

$$\frac{M_t}{P_t} = \frac{C_t}{\overline{v}} = \frac{Y_t}{\overline{v}\left(1+\frac{\gamma\overline{v}}{1+\overline{v}}\right)}. \tag{19}$$

This implies that real money balances are bounded because Y_t is assumed to be bounded. For B_t/M_t to increase without bound, it must be that B_t/P_t must grow ad infinitum. But that cannot be an equilibrium by the now familiar argument that agents would violate transversality by holding infinite amounts of an asset. They could increase utility arbitrarily by selling off some part of it.

Suppose B_t/M_t falls. It cannot fall below zero as both components are constrained to be positive. However, it could be the case that it falls until $B_t = 0$. From the first order condition for consols, one sees that $\mu_t > 0$ implies that $v_t > \overline{v}$. This in turn implies that $z_t < \overline{z}$. For the rest of the argument, see Sims' "Simple Model".

Knowing that $B_t/M_t = B/M$, one can return to the government budget constraint, before taking expectations:

$$\frac{B_t}{M_t r_t} = \frac{1+r_t}{r_t}\frac{B_{t-1}}{M_{t-1}}\frac{M_{t-1}}{M_t} - 1 + \frac{M_{t-1}}{M_t} - \overline{\tau}v_t\frac{\left(1+\frac{\gamma v_t}{1+v_t}\right)}{Y_t}. \tag{20}$$

The corresponding equation for the one-period bond model is

$$\frac{B_t}{M_t} = R_{t-1}\frac{B_{t-1}}{M_{t-1}}\frac{M_{t-1}}{M_t} - 1 + \frac{M_{t-1}}{M_t} - \overline{\tau} v_t \frac{\left(1 + \frac{\gamma v_t}{1+v_t}\right)}{Y_1} . \tag{21}$$

Rearranging (20) and (20) and using the fact that at t (but possibly not, in the first period, $t-1$) the $r_t = \overline{r}$ or $R_t = \overline{R}$ policy is permanently in place, we arrive at

$$\frac{M_{t-1}}{M_t}\left(1 + (1+\overline{r})\frac{B_{t-1}}{M_{t-1}\overline{r}}\right) = 1 + \frac{B_t}{M_t\overline{r}} + \overline{\tau v}\frac{\left(1 + \frac{\gamma\overline{v}}{1+\overline{v}}\right)}{Y_t} \tag{22}$$

and

$$\frac{M_{t-1}}{M_t}\left(1 + R_{t-1}\frac{B_{t-1}}{M_{t-1}}\right) = 1 + \frac{B_t}{M_t} + \overline{\tau v}\frac{\left(1 + \frac{\gamma\overline{v}}{1+\overline{v}}\right)}{Y_t} , \tag{23}$$

respectively, as in the paper. Given M_{t-1}, and the realization of Y_t, this implies a path for M_t and hence a path for P_t, by

$$P_t = M_t \frac{\overline{v}\left(1 + \frac{\gamma\overline{v}}{1+\overline{v}}\right)}{Y_t} \tag{24}$$

The arguments for existence and uniqueness are thus the same as in the one period bond case.

1.5. **Question 3.** Show that if we start in an equilibrium with constant interest rate and taxes, chosen so that there is no trend in prices, an unanticipated change to a new policy with same tax level, but a lower nominal interest rate, produces different time paths for prices depending on whether one is in the consol or one-period bond case, even if the new interest rate is such that long run inflation is the same for both types of economies.

1.6. **Answer.** We can refer to (13) and (14). We have already argued that, so long as the new lower interest rate is the same for both short-bond and consol models, the right-hand sides of these two equations are the same. However the coefficient in parentheses that multiplies the money growth rate on the left-hand side is different in the two models in the first period after the switch to the lower interest rate. This already answers the question as posed. However it is interesting to go further and observe first that after the initial period, the two equations are the same, so that the money growth rate, and hence the rate of inflation, will be the same in both models. In particular, the expected money growth rate is lower with lower $\overline{r}$. We can see this directly from (22) and (23).

To see the initial effect on prices of the "monetary expansion" (the r or R decrease) we multiply (13) and (14) by $v_{t-1}/\overline{v}$, to convert M_t/M_{t-1} to $P_tC_t/(P_{t-1}C_{t-1}$. The result is

$$\frac{P_{t-1}C_{t-1}}{P_tC_t}\left(1+(1+\bar{r})\frac{B_{t-1}}{M_{t-1}\bar{r}}\right)=\frac{v_{t-1}}{\bar{v}}+\frac{B_tv_{t-1}}{v_tM_t\bar{r}}+v_{t-1}\overline{\tau}\frac{\left(1+\frac{\gamma\bar{v}}{1+\bar{v}}\right)}{Y_t} \tag{25}$$

$$\frac{P_{t-1}C_{t-1}}{P_tC_t}\left(1+R_{t-1}\frac{B_{t-1}}{M_{t-1}}\right)=\frac{v_{t-1}}{\bar{v}}+\frac{B_tv_{t-1}}{v_tM_t}+v_{t-1}\overline{\tau}\frac{\left(1+\frac{\gamma\bar{v}}{1+\bar{v}}\right)}{Y_t}\,. \tag{26}$$

The right-hand side in these two equations has the same value by our previous arguments. The first two terms on the right-hand side are increased relative to the equilibrium values before the monetary expansion. The last term on the right will be slightly decreased, but so long as transactions costs are a small part of Y, the influence of this last term will be small and the overall effect of the expansion will be to raise the right-hand side. In (26), the short-bonds case, this can be seen unambiguously to imply that P_t in the first period of the new policy will be *lower* than it would have been in the original equilibrium. I.e., the "expansionary" monetary policy not only produces lower expected money growth and expected inflation, it makes the initial price level jump downward. Or to put the matter another way, the only way to produce a decline in the interest rate is to undertake a deflationary policy that contracts M.

The situation is different with (25), the consol-debt case. There, because the coefficient in parentheses on the left increases as $\bar{r}$ increases, the required decline in initial P will certainly be smaller than in the case of short debt, and it can easily be, when interest-bearing debt is large in value relative to non-interest-bearing debt, that the decline in $\bar{r}$ produces an initial rise in P. This occurs because the decline in $\bar{r}$ produces a capital gain for bond-holders at the initial price level; even though the higher anticipated future seignorage revenue increases the equilibrium real value of the debt, the drop in r is likely to produce an increase in the value of the debt that exceeds the equilibrium increase, so prices must rise to compensate.

This result is of some interest, because it suggests that in the presence of a fiscal policy that is unresponsive to the level of the debt, interest rate policy has the usual inflationary or deflationary impacts only if there is substantial long-term debt outstanding.