1. Answer to "Effects of Taxes in a Stochastic Growth Model" from Spring 98 Comps

(a) Letting λ be the Lagrange multiplier on (2), μ that on (3), and θ that on (4), the Euler equation FOC's are

$$\partial C: \qquad C_t^{-\gamma} = \lambda_t \cdot (1+\tau_t) \tag{1}$$

$$\partial K: \qquad \mu_t = \beta E_t \left[\delta\mu_{t+1} + \lambda_{t+1} \cdot \left((1-\nu_{t+1})\alpha A_{t+1} K_t^{\alpha-1} + \phi \frac{I_{t+1}^2}{K_t^2} \right) \right] \tag{2}$$

$$\partial I: \qquad \lambda_t \left(1 + 2\phi \frac{I_t}{K_{t-1}} \right) = \mu_t \tag{3}$$

$$\partial B: \qquad \frac{\lambda_t}{Pt} - \theta_t = \beta R_t E_t \left[\frac{\lambda_{t+1}}{P_{t+1}} \right] . \tag{4}$$

(b) Subtracting the government budget constraint (GBC) (5) from the indivdual agent constraint (2) gives us the social resource constraint (SRC)

$$C_t + I_t \cdot \left(1 + \phi \frac{I_t}{K_{t-1}} \right) = A_t K_{t-1}^{\alpha} \, , \tag{5}$$

which does not depend on the tax parameters or on B or P. Thus it is certainly feasible to have the same $\{C, I, K\}$ allocation under all three tax and debt regimes, but we still need to check that agents can perceive such allocations as satisfying their optimization problems.

If we use the ∂C FOC to solve for λ and the ∂I FOC to solve for μ, then substitute these expressions in the ∂K FOC, we arrive at

$$\left(1 + \frac{2\phi I_t}{K_{t-1}} \right) \frac{C_t^{-\gamma}}{1+\tau_t} = \beta E_t \left[\frac{C_{t+1}^{-\gamma}}{(1+\tau_{t+1})} \left(1 + 2\delta\phi \frac{I_{t+1}}{K_t} + \left((1-\nu_{t+1})\alpha A_{t+1} K_t^{\alpha-1} + \phi \frac{I_{t+1}^2}{K_t^2} \right) \right) \right] \tag{6}$$

Notice that if ν is always zero and if τ is constant, all tax terms disappear from (6), so certainly if there is an equilibrium with $\nu = \tau = 0$, the same real allocation will satisfy the agent FOC's when instead $\nu = 0$ and $\tau_t = \bar{\tau}$. All that remains is to check that the government budget constraint can still be satisfied. With $\nu = 0$ and $\tau_t = \bar{\tau}$, we can multiply the GBC by $\lambda_t = C_t^{-\gamma}$, and take E_{t-1} of it to arrive at

$$E_t \left[\frac{B_t}{P_t C_t^{-\gamma}} \right] = \beta^{-1} \frac{B_{t-1}}{P_{t-1} C_{t-1}} - \bar{\tau} E_t [C_t^{1-\gamma}] \tag{7}$$

This is an unstable equation in $E_t[B_s C_s^{-\gamma}/P_s]$. It does have a stable solution, however – that obtained by solving forward to express $B_t C_t^{-\gamma}/P_t$ as a discounted sum of expected future values of $E_t[\bar{\tau} C_{t+s}^{1-\gamma}]$. So given the C sequence from the no-tax equilibrium and an initial positive value for B, we can find an initial P that keeps real debt non-explosive. This same process can be repeated at each t to generate the complete solution for the P process.

Of course in the reverse case of $\tau_t = 0$, $\nu_t = \bar{\nu} > 0$, the FOC (6) is not satisfied at the no-tax solution values for the real variables.

(c) Again referring to (6), we can see that cancelling out τ required that τ_t be constant for two reasons. First, even if τ_{t+1} were independent of the rest of the right-hand side of (6) so that it factored out, and even if its expectation were constant, the τ_t on the left-hand side is a random variable that will fluctuate, preventing cancellation. But also, since it was given that τ_t depends on A_t, it will be correlated with the rest of the right-hand side of the equation and cannot be factored out. This is not actually an argument that the two equilibria can never be equal. It only shows that they will not be equal in general. A good answer would simply explain why the τ's don't automatically cancel out in this case.

2. Answer to "Price level determinacy with long term government debt" from Fall 98 comps.

(a) The Euler equations are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t \tag{8}$$

$$\partial B: \qquad \frac{\lambda_t}{P_t R_t} = \beta E_t \left[\frac{\lambda_{t+1}(1 + R_{t+1})}{P_{t+1} R_{t+1}} \right] + \mu_t \, . \tag{9}$$

Here μ_t is the Lagrange multiplier on the $B \geq 0$ constraint, 0 unless the constraint binds. The first order conditions also include a transversality condition, which here is

$$\limsup_{T \to \infty} \left\{ \beta^T \frac{\lambda_T dB_T}{P_T R_T} \right\} \geq 0 \, , \tag{10}$$

where dB_T is any feasible deviation from the optimal path for B_T. Students are expected to get the Euler equations right and to at least mention transversality.

(b) The Euler equations imply

$$\frac{1}{R_t P_t C_t} = \beta E_t \left[\frac{(1 + R_{t+1})}{R_{t+1} P_{t+1} C_{t+1}} \right] \, . \tag{11}$$

If we move the GBC one period forward in time, divide it by C_{t+1}, and take E_t of it, we can use (11) and the policy rules $R_t \equiv \bar{R}$, $\tau \equiv \bar{\tau}$ to simplify to

$$\beta^{-1} \frac{B_t}{P_t C_t} = E_t \left[\frac{B_{t+1}}{P_{t+1} C_{t+1}} + \frac{\bar{R} \bar{\tau}}{C_{t+1}} \right] \, . \tag{12}$$

Note that the social resource constraint, derivable from the GBC and the private budget constraint, requires $C_t \equiv Y_t$, and Y_t was assumed i.i.d. Therefore the last term on the right of (12) is just a constant. The equation is then clearly an unstable difference equation in $E_t[B_{t+s}/(P_{t+1}C_{t+s})]$, with the unique stable solution

$$\frac{B_t}{P_tC_t} = \frac{\bar{\tau}\bar{R}}{\beta^{-1}-1} E\left[\frac{1}{Y_t}\right] . \tag{13}$$

Using this relationship in the GBC, we get an equation containing only exogenous Y_t, policy constants, the predetermined B_{t-1}, and P_t. We can solve this at each t for the unique price level consistent with equilibrium. Other values of P would produce unstable solutions to (12), and these would violate the transversality conditions of the agents, or the $B_t \geq 0$ constraint.

(c) It may be tempting to stop here and claim that, with R in the numerator on the right-hand side of (13), and P in the denominator on the left, we can see that increasing R lowers P. However, the argument is not quite that simple, because B as well as P will respond within the period to the surprise change in R. Let κ be the constant defined by the right-hand side of (13), divided by $\bar{R}$. (We divide by $\bar{R}$ so that κ will be constant not only over time, but also across equilibria with different $\bar{R}$ values. Then the GBC, divided through by C_t, is given by

$$\kappa + \frac{\bar{\tau}}{C_t} = \frac{\bar{R}+1}{\bar{R}} \frac{B_{t-1}}{P_tC_t} . \tag{14}$$

Rearranging again, we arrive at

$$\frac{(\kappa + \bar{\tau}/C_t)\bar{R}}{1+\bar{R}} = \frac{B_{t-1}}{P_tC_t} . \tag{15}$$

The right-hand side of (15) is monotone decreasing in P_t. The left-hand side is monotone increasing in $\bar{R}$. So the initial effect of a surprise, permanent increase in $\bar{R}$ is unambiguous: it lowers the price level. Of course the inflation rate between $t-1$ and t is affected in the same direction as P_t by changes at t in $\bar{R}$. The expected rate of growth of nominal income from t to $t+1$, after $\bar{R}$ has been set at t, is determined by (11), which we can write (using the SRC) as

$$\frac{1}{\beta(1+\bar{R})} = E_t\left[\frac{P_tY_t}{P_{t+1}Y_{t+1}}\right] . \tag{16}$$

This shows that expected nominal income growth is positively related to $\bar{R}$. Since Y is exogenous, the same will be true of expected price inflation, though an exact formula would require fully solving the model.