Problem Set #3, Suggested Solutions

i) First, it is easy to show

$$\frac{\partial^2}{\partial c_t^{\,2}}\left(1-e^{-\gamma c_t}\right)=-\gamma^2 e^{-\gamma c_t}<0$$
.

Therefore, the one-period utility function is concave. Since it is also bounded on the positive consumption path, its discounted sum is well defined. In other words, $\sum_{t=0}^{\infty}\beta^t\left(1-e^{-\gamma c_t}\right)$ is well defined so long as $0<\beta<1$.

Since the above function is a pointwise limit of linear combinations of concave functions with positive weights, the above is also concave. Finally, since the expectation operator is linear, it keeps the concavity, too. This proves that the objective function is concave over positive sequences of consumption.

ii) Conditions (2) through (4) define subsets of the space of sequences. The intersection of all these subsets is the feasible set. Intersections of convex sets are convex. Conditions (3) and (4) define sets by linear equality or inequality constraints, which are automatically convex. We are going to show the convexity of the set which satisfies (2) for a given *t*.

First, we need to assume $\phi>0$ and $0<\alpha<1$, $A_t\geq 0$ for all *t*.

Generally, if a function f(x, y, z) is concave, the set {x,y,z} which satisfies f(x, y, z)>0 is convex(this is the definition of the quasi-concavity. Concave functions are always quasi-concave). Therefore, we are going to show the concavity of

$$f(x,y,z)=A_t x^{\alpha}-y-\phi\frac{y^2}{x}-z$$
.

The Hessian Matrix of this is

$$H=\begin{bmatrix} -\alpha\cdot(1-\alpha)A_t x^{\alpha}-2\phi\dfrac{y^2}{x^3} & 2\phi\dfrac{y}{x^2} & 0 \\ 2\phi\dfrac{y}{x^2} & \dfrac{-2\phi}{x} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this is negative semi-definite, f is concave. Having showed that all the individual sets defined by (2)-(4) are convex, we are now finished with showing that there intersection is convex.

iii) Set up the following Lagrangian:

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\left(1-e^{-\gamma c_t}\right)+\sum_{t=0}^{\infty}\beta^t\lambda_t\left(A_t k_{t-1}^{\ \alpha}-c_t-I_t\left(1+\phi\frac{I_t}{k_{t-1}}\right)\right)+\sum_{t=0}^{\infty}\beta^t\mu_t\left(I_t+\delta k_{t-1}-k_t\right)\right].$$

The F.O.C.s are the following:

$$E_t\left[\beta\lambda_{t+1}\cdot\left(\phi\frac{I_{t+1}^2}{K_t}+A_{t+1}\alpha K_t^{\alpha-1}\right)-\mu_t+\beta\delta\mu_{t+1}\right]=0$$

$$\lambda_t\left(1+\phi\frac{2I_t}{k_{t-1}}\right)=\mu_t,$$

$$\lambda_t=\gamma e^{-\gamma c_t},$$

$$A_t k_{t-1}{}^{\alpha}-c_t-I_t\left(1+\phi\frac{I_t}{k_{t-1}}\right)=0,$$

$$I_t+\delta k_{t-1}-k_t=0.$$

Transversality in this model requires that

$$E\left[\beta^t\mu_t K_t\right]=E\left[\beta^t\gamma e^{-\gamma C_t}\left(1+\phi\frac{2I_t}{K_{t-1}}\right)K_t\right]\to 0\,.$$

With $\delta<1$, and $C\geq 0$, the technology implies that there is an upper bound on *K*, and hence *C*, when there is an upper bound on *A*. Thus it is likely that transversality itself puts no constraints on the explosiveness of the solution – any feasible solution automatically fails to explode. (It is possible that there could be solutions collapsing exponentially to zero that have to be ruled out by transversality – I haven't checked). But the infeasibility of unbounded solutions itself suggests that explosive solutions to the linearized models should be ruled out.

The deterministic steady state can be calculated by assuming *A*=1, and *K*, *C* and *I* are constant over time.

$$\bar{k}=\left(\frac{(1+2\phi(1-\delta))(1-\delta\beta)-\phi\beta(1-\delta)^2}{A\alpha\beta}\right)^{\frac{1}{\alpha-1}},$$

$$\bar{I}=(1-\delta)\bar{k},$$

$$\bar{c}=A\bar{k}^{\alpha}-\bar{I}(1+(1-\delta)\phi),$$

$$\bar{\lambda}=\gamma e^{-\gamma\bar{c}},$$

$$\bar{\mu}=\bar{\lambda}(1+2(1-\delta)\phi).$$

iv)

Notice that if you define dx as

$$dx=\log x-\log\bar{x},$$

then, the log linear approximation of f(x) around $\bar{x}$ is f($\bar{x}$) + f'($\bar{x}$)$\bar{x}$dx. We are going to take a log linear approximations of the F.O.C.s.

$$\gamma\bar{c}dc_t + d\lambda_t = 0,$$

$$d\lambda_t - d\mu_t + \frac{2\phi(1-\delta)}{(1+2(1-\delta)\phi)}dI_t = \frac{2\phi(1-\delta)}{(1+2(1-\delta)\phi)}dk_{t-1},$$

$$\beta\bar{\lambda}\left[\phi(1-\delta)^2 + \alpha A\bar{k}^{\alpha-1}\right]d\lambda_t + 2\beta\bar{\lambda}\phi(1-\delta)^2 dI_t + \beta\delta\bar{\mu}d\mu_t$$
$$= \beta\bar{\lambda}\left[\alpha(1-\alpha)A\bar{k}^{\alpha-1} + 2\phi(1-\delta)^2\right]dk_{t-1} + \bar{\mu}d\mu_{t-1} - \beta\bar{\lambda}\alpha A\bar{k}^{\alpha-1}dA_t + d\varepsilon_t,$$

$$\bar{c}dc_t + (1-\delta)\bar{k}\left[1+2\phi(1-\delta)\right]dI_t = \left[\phi(1-\delta)^2\bar{k} + \alpha A\bar{k}^{\alpha}\right]dk_{t-1} + A\bar{k}^{\alpha}dA_t,$$

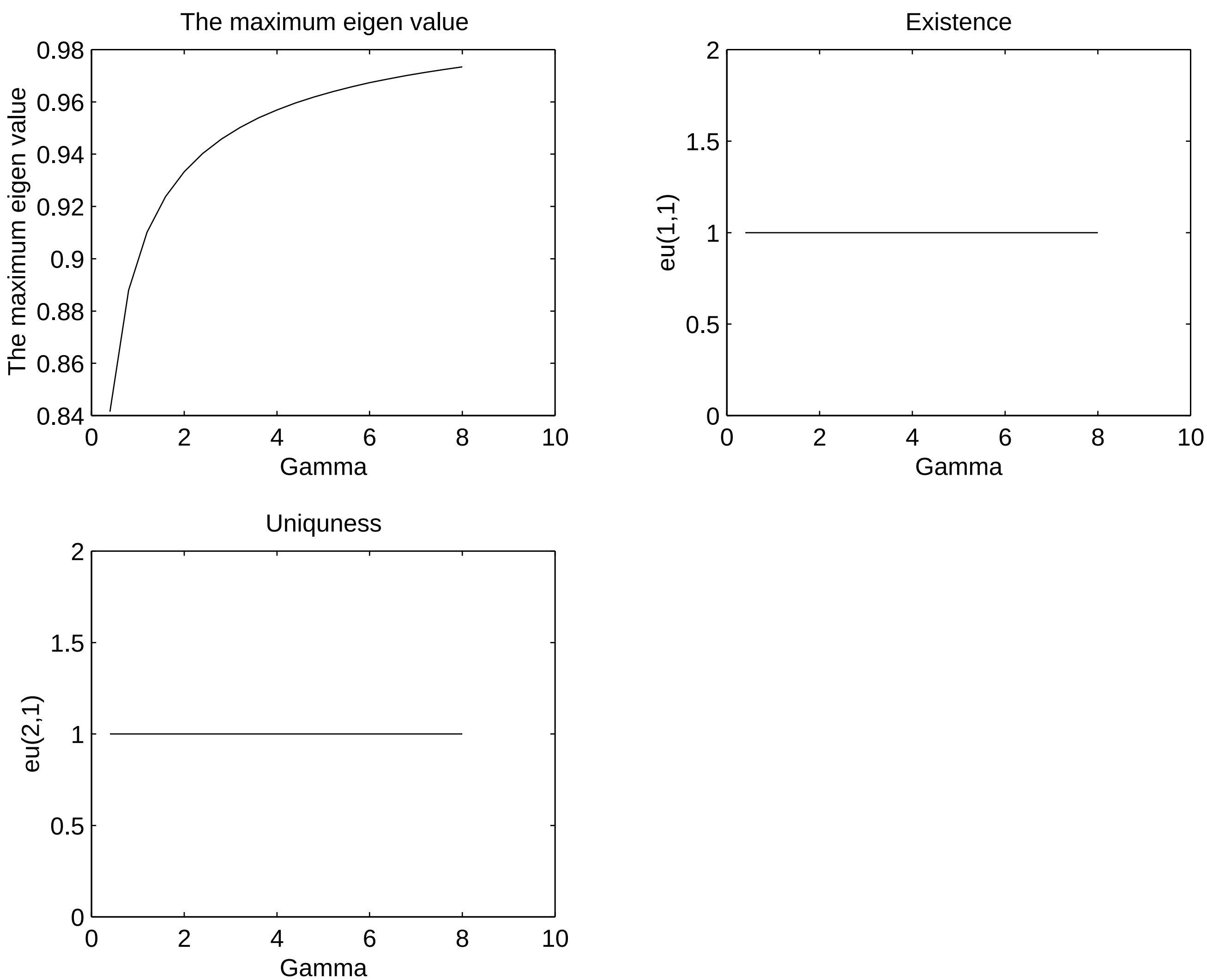
$$dk_t - (1-\delta)dI_t = \delta dk_{t-1}.$$

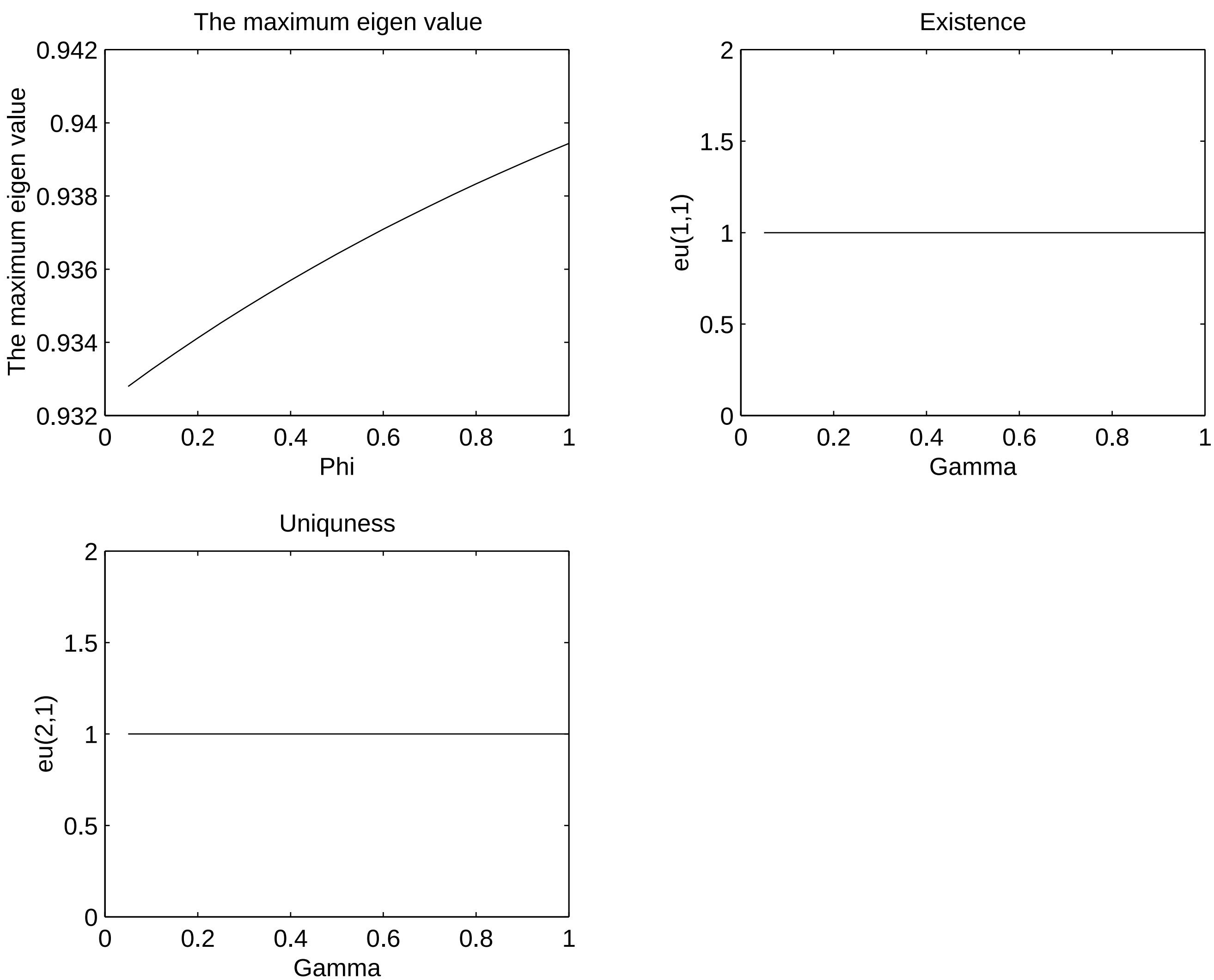
Therefore,

$$y_t = \begin{pmatrix} dc_t \\ dI_t \\ dk_t \\ d\lambda_t \\ d\mu_t \end{pmatrix}, \qquad \Gamma_0 = \begin{pmatrix} \gamma\bar{c} & 0 & 0 & 1 & 0 \\ 0 & \frac{2\phi(1-\delta)}{(1+2(1-\delta)\phi)} & 0 & 1 & -1 \\ 0 & 2\beta\bar{\lambda}\phi(1-\delta)^2 & 0 & \beta\bar{\lambda}\left[\phi(1-\delta)^2 + \alpha A\bar{k}^{\alpha-1}\right] & \beta\delta\bar{\mu} \\ \bar{c} & (1-\delta)\bar{k}\left[1+2\phi(1-\delta)\right] & 0 & 0 & 0 \\ 0 & -(1-\delta) & 1 & 0 & 0 \end{pmatrix},$$

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2\phi(1-\delta)}{(1+2(1-\delta)\phi)} & 0 & 0 \\ 0 & 0 & \beta\bar{\lambda}\left[\alpha(1-\alpha)A\bar{k}^{\alpha-1} + 2\phi(1-\delta)^2\right] & 0 & \bar{\mu} \\ 0 & 0 & \left[\phi(1-\delta)^2\bar{k} + \alpha A\bar{k}^{\alpha}\right] & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} 0 \\ 0 \\ -\beta\bar{\lambda}\alpha A\bar{k}^{\alpha-1} \\ A\bar{k}^{\alpha} \\ 0 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Two matlab files, ps31.m and ps32.m, that produce answers to this section of the problem set, are available on the web site. The plots they produce are shown below. They show that at the parameters settings we consider, the maximal root stays between .93 and .95 for the ϕ variation and goes just above .95 for very large values of γ. For γ's in the range often considered natural, around 2, the roots are substantially below .95. These deviations are big enough to be statistically noticeable.


The maximum eigen value
The maximum eigen value
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0.84
0
2
4
6
8
10
Gamma
Existence
eu(1,1)
2
1.5
1
0.5
0
0
2
4
6
8
10
Gamma
Uniquness
eu(2,1)
2
1.5
1
0.5
0
0
2
4
6
8
10
Gamma


The maximum eigen value
The maximum eigen value
0.942
0.94
0.938
0.936
0.934
0.932
0
0.2
0.4
0.6
0.8
1
Phi
Existence
eu(1,1)
2
1.5
1
0.5
0
Gamma
Uniquness
eu(2,1)
Gamma