Answer for 4/15 Exercise Problem 1

(Problem 2 answers were discussed completely in section meeting.)

1. For the case of $f(v) = f_N(v) = v^2$, the analysis can exactly parallel that in "A Simple Model …" for the case $f(v) = f_U(v) = v$. One first checks that

$$z_t = \frac{\lambda_t}{P_t} = \frac{1}{M_t v_t \cdot \left(1 + \gamma f_t' v_t + \gamma f_t\right)} = \frac{1}{\overline{M} v_t \cdot \left(1 + 3\gamma v_t^2\right)} \qquad [1]$$

is monotone in *v*, which it obviously is. Then one considers whether

$$1 - \gamma f_t' v_t^2 = 1 - 2\gamma v_t^3 = \beta \qquad [2]$$

has a unique root, corresponding to steady-state *v*. Clearly in this case it does have a single root for any β between 0 and 1. Then finally one considers whether the difference equation

$$z_t \cdot \left(1 - \gamma f_t' v_t^2\right) = \beta E_t z_{t+1} \qquad [3]$$

has only the steady-state as a stable solution. Because of the monotonicity of *z* and $1 - \gamma f' v^2 = 1 - 2\gamma v^3$ as functions of *v*, this is also easily verified. Then transversality and feasibility rule out any solution to the FOC's other than the one with constant *v*.

For the case $f(v) = f_X(v) = v^2 \big/ \left(1 + v^2\right)$, these steps are not parallel to those in the "Simple Model…" paper. Here

$$z = \frac{1}{\overline{M} v \cdot \left(1 + 3\gamma v^2 \big/ \left(1 + v^2\right)^2\right)} \ . \qquad [4]$$

That this is monotone decreasing in *v* is not immediately obvious, but it is easy to plot it with Matlab or Gauss and it can be proved via somewhat tedious algebra that it is monotone decreasing for any positive γ.

The tedious algebra

$$\frac{d(1/z)}{dv} = \overline{M} \cdot \left(1 + \frac{9\gamma v^2}{\left(1 + v^2\right)^2} - \frac{12\gamma v^4}{\left(1 + v^2\right)^3}\right). \qquad [5]$$

This is equal to $\overline{M}$ at $v = 0$ and also in the limit as $v \to \infty$. We can locate its extrema by differentiating again, obtaining

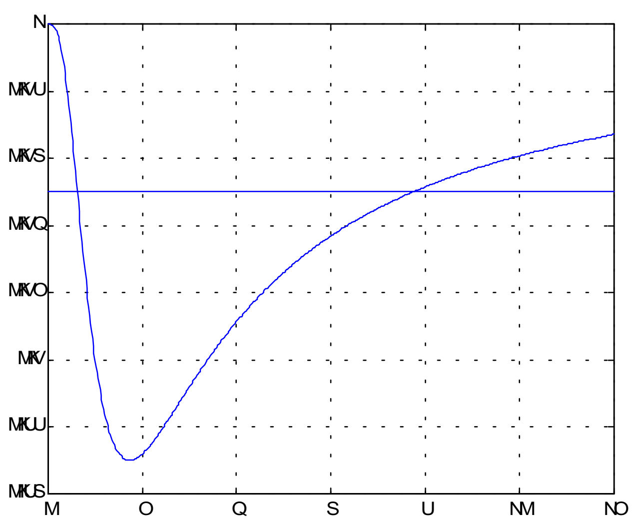
$$\begin{aligned}\frac{d^2(1/z)}{dv^2} &= \frac{18\gamma v}{\left(1+v^2\right)^2} - \frac{36\gamma v^3}{\left(1+v^2\right)^3} - \frac{48\gamma v^3}{\left(1+v^2\right)^3} + \frac{72\gamma v^5}{\left(1+v^2\right)^4} \\ &= \frac{2v\gamma}{\left(1+v^2\right)^4}\cdot\left(9\cdot\left(1+v^2\right)^2 - 42v^2\cdot\left(1+v^2\right) + 36v^4\right). \\ &= \frac{6v\gamma}{\left(1+v^2\right)^4}\cdot\left(3-8v^2-2v^4\right)\end{aligned} \qquad [6]$$

The last parenthesis in [2] contains a quadratic form in v^2, which can easily be determined to have two real roots, one positive and one (impossibly) negative. Since the expression is positive at $v=0$ and negative for large *v*, we know that it is everywhere greater than or equal to its value at zero and ∞, i.e. 1. Therefore *z* is decreasing in *v* for all positive *v*.

We need also to consider the coefficient on the left-hand side of [3], which here is

$$1-\gamma f' v^2 = 1 - \frac{2\gamma v}{\left(1+v^2\right)^2}\,. \qquad [7]$$

This expression is not monotone in *v*. It is decreasing in *v* for *v* near zero, but then is increasing in *v*, converging to 1 from below, for large values of *v*. It may therefore remain always above β, implying that there is no steady state for the model, or be equal to β for two distinct values of *v*, implying the existence of two possible steady-state *v*'s. (There is also a knife-edge special case of a single steady-state *v*.) For example, with $\gamma = .2$, $\beta = .95$, the situation looks like this:



Near the lower steady state, since [7] is decreasing in *v*, the difference equation [3] is unstable. But near the upper steady state it is stable. Thus the argument that price levels above that consistent with the lower steady state value of *v* are impossible, because the would lead eventually to arbitrarily high *v* and hence violation of FOC's, does not work here. It appears that any initial price level above that consistent with the lower steady-state value of *v* is consistent with equilibrium, because every such initial price leads to v converging toward, and eventually fluctuating around, the steady state level.

This is the best answer I expected from students in the course, and is already more complicated than you could be expected to deal with in a time-constrained exam, unless a very similar model had been gone over in detail in notes or an exercise. (Of course, now this model has been gone over in an exercise.)

However, there is a further issue to consider. With $f = f_X$, or indeed even with $f = f_B$ from the "Simple Model …" paper, the consumer's budget constraint, with everything moved to the left-hand-side, is not a convex function of the consumer's choice variables. This raises the question of whether the FOC's actually define a maximum. In the case of $f = f_B$, the constraint function is globally quasi-concave, so that the constraint set itself is convex and the FOC's must indeed define a maximum. To check quasi-concavity, we observe that the budget constraint is linear in the choice variables B_t, M_{t-1} and B_{t-1}, which implies that it is quasi-concave iff it is quasi-concave jointly in M_t, C_t. Using implicit differentiation, we can find the derivative of M_t with respect to C_t along the boundary of the budget set as

$$-\frac{1+\gamma f+\gamma f' v}{1/P-\gamma f' v^2/P} \quad . \qquad [8]$$

The second derivative is then

$$\begin{aligned} & -\frac{P\cdot(2\gamma f'+\gamma f'' v)}{1-\gamma f' v^2}-\frac{P\cdot\left(\gamma f'' v^2+2\gamma f' v\right)(1+\gamma f+\gamma f' v)}{\left(1-\gamma f' v^2\right)^2} \\ & =-P\gamma\frac{(2f'+f''v)\left(1-\gamma f' v^2+v+\gamma f v+\gamma f' v^2\right)}{\left(1-\gamma f' v^2\right)} \\ & =-P\gamma\frac{(2f'+f''v)(1+v+\gamma f v)}{\left(1-\gamma f' v^2\right)} \end{aligned} \quad . \qquad [9]$$

For concavity of the constraint set, we need this to be negative. Obviously, so long as we have not violated the FOC's by making $1-\gamma f' v^2$ negative, the sign of the last term in [9] is determined by that of $2f'+f''v$. It is easy to check that the second derivative is negative, then, for the $f = f_B$ case. For the case at hand, $f = f_X$, we find

$$2f'+f''v=\frac{4v}{\left(1+v^2\right)^2}+\frac{2v}{\left(1+v^2\right)^2}-\frac{8v^3}{\left(1+v^2\right)^3}=\frac{6v-2v^3}{\left(1+v^2\right)^3} \, . \qquad [10]$$

This expression changes sign as v increases above $\sqrt{3}$. We can see from the diagram above that the lower steady state is within the region where the constraint sets are convex with the parameters chosen there, but the upper steady state is not. The fact that the constraint set is not convex does not prove that the FOC's do not define an optimum – indeed I suspect that they do still define an optimum. However we can no longer be sure that solutions to the FOC's define an optimum in this case. One check is to verify that there are not higher values of the objective function obtainable by letting $v \to \infty$, i.e. (for the consumer) by letting M go to zero. At any date t, the consumer can obtain an increase in utility of $\log(C + M/P) - \log C = \log(1 + 1/v)$ in the current period by consuming his real balances. The corresponding loss in utility due to increased transactions costs is $-\log\left(1 + \gamma v^2 \big/ (1 + v^2)\right)$ per period, which as a discounted stream gets multiplied by $1/(1-\beta)$. For the parameter values underlying the figure above, there is a gain in utility from setting M to zero only at values of v below the lower steady-state, which were ruled out by transversality in any case. This is not a complete proof that the FOC's define an optimum, and discussion of these particular parameter values is obviously not a complete analysis of the second-order conditions. But since these notes have already gone much farther into these thickets than you were expected to go, we will stop here.

The analysis of the second-order conditions for this problem is far too complicated algebraically for an exam question. However the point that convexity of constraint sets is an issue, because without convexity we cannot be sure that FOC's define an optimum, is something you are expected to be aware of.