Econ. 511b Spring 1997 C. Sims

Sticky Price Answers

3/30: This version provides a complete answer and corrects some errors both in the problem statement and in the previous partial versions of this answer sheet. The most important error in the problem statement, previously announced, was the change to elastic labor supply in problem 4. The problem also should not have used such a large change in M as the "shock" in problem 4, as this size shock is clearly far too large to allow the linearization to work accurately. In one previous version of the solution, q=1 was suggested as a parameter value in 4. This won't work. The model, like most Dixit-Stiglitz setups, requires q>1. Algebra errors for the answer to problem 4 occurred in all previous versions of this answer sheet, and could even still be present, though the answer now at least looks right.

1. and 2.: The firm in this problem does not have a dynamic problem. It chooses only L, and the FOC w.r.t. L gives us simply

$$aL^{a-1} = \frac{W}{P} \ , \tag{1}$$

i.e. real wage equals marginal product. For the individual, the FOC's are

¶C: $$\frac{1}{C} = l \cdot (1 + 2gV) \tag{2}$$

¶B: $$\frac{-\dot{l}}{P} + \frac{l\dot{P}}{P^2} + \frac{b\,l}{P} = \frac{r\,l}{P} \tag{3}$$

¶M: $$-\frac{lPC^2}{M^2} - \frac{\dot{l}}{P} + \frac{l\dot{P}}{P^2} + \frac{b\,l}{P} = 0 \ . \tag{4}$$

Note that FOC's are in exactly the same form as in the model we have already discussed that does not involve labor (or stickiness). As in our previous discussions, we can use (2) to eliminate l from the system, and then deduce

$$gV^2 = r \tag{5}$$

$$\frac{\dot{C}}{C} + \frac{2g\dot{V}}{(1+2gV)} = gV^2 - \frac{\dot{P}}{P} - b \ . \tag{6}$$

With the $M \equiv \overline{M}$ policy, $\dot{V}/V = \dot{P}/P + \dot{C}/C$, so that (6) becomes the one-variable differential equation

$$\frac{\dot{V}}{V} + \frac{2g\dot{V}}{1+2gV} = gV^2 - b \ . \tag{7}$$

As we noted before, this is an unstable differential equation with a unique stable solution: $V \equiv \overline{V} = \sqrt{b/g}$. It is also easy to check numerically (e.g. with Matlab's ode23 or ode45 programs) that when $V > \bar{V}$, V goes to infinity in finite time, while if $V < \bar{V}$, V goes to zero eventually. Just as before, we can argue that individuals could not see paths on which V becomes arbitrarily small as optimal. We can also introduce assumptions about policy – e.g. that a backstop fiscal policy guarantees a maximum on P – that eliminate the paths that make V go to infinity. We then conclude that any equilibrium must have V fixed at $\bar{V}$.

The social resource constraint, derived from the GBC and the firm's and individual's budget constraints, is

$$C \cdot \left(1 + g\bar{V}\right) = L^a . \tag{8}$$

Then the constancy of velocity implies

$$\frac{PC}{\bar{M}} = \frac{PL^a}{\bar{M} \cdot \left(1 + g\bar{V}\right)} = \bar{V} . \tag{9}$$

We also know from the firm FOC that

$$P = \frac{WL^{1-a}}{a} , \tag{10}$$

so that from (9) and (10)

$$WL = a\bar{M} \cdot \left(1 + g\bar{V}\right)\bar{V} . \tag{11}$$

Thus the nominal wage bill is constant, and $\dot{W}/W = -\dot{L}/L$. Using this fact, and writing ℓ for $\log L$, we can conclude from the wage adjustment equation that

$$\dot{\ell} = -q\ell . \tag{12}$$

So in the steady state equilibrium, $\ell = 0$, $W = a\bar{M} \cdot \left(1 + g\bar{V}\right)\bar{V}$, $C = 1/\left(1 + g\bar{V}\right)$, $P = W/a$. Because of our assumption that W changes only in accord with the wage adjustment differential equation, it cannot jump in response to the surprise increase in M. From (11) we see that this means that L initially jumps in proportion to the jump in $\bar{M}$, i.e. by a factor of 2. From (10) we see that this implies also an initial upward jump in P, but by a factor 2^{1-a}. From (8) we see that C jumps upward by the factor 2^a. From this initial point, all variables move monotonely toward the steady state equilibrium. From (12), log$L$ and log$W$ return toward steady-state values at the exponential rate q. The variables log$P$ and log$C$ then return toward their steady state values at the exponential rate aq. So long as fiscal policy makes taxes respond to increased debt strongly enough to keep the time path of real debt bounded, there is no influence of fiscal policy on the time paths of P, C, L, or W.

3. Because the firm's problem is static in question 2, it makes no difference how it discounts profits. The only contribution of the firm's problem to the solution is the condition (10), which does not involve the discount rate. In question 4 below, the firm has the dynamic problem of deciding the path of price changes, so discounting matters. The question is worded badly, though, in that it refers to a "planner's problem". We already know that in these models, where producing real balances is costless, there is no social planner's optimum, because increasing real

balances will always increase welfare. What the problem should have done is ask how discounting should be done by the firm if the decentralized problem, with firm and consumer, is to deliver the same solution we would get if the firm and individual problems were merged. For this it is essential that the firm see the same set of transformation rates between dividends p_t at different dates as does the individual for transformation rates between C_t at different dates. With the individual holding bonds, she is trading dollars at different dates according to the interest rate. Thus if the firm discounts dollars – nominal profits – at the interest rate, the firm's and individual's decisions are properly coordinated. But in this model consumption varies, so the *real* interest rate is non-constant in equilibrium. Thus individuals' and firms' decisions are not properly coordinated if the firm discounts real dividends at the fixed rate b.

4. This problem had to be restated, to include leisure in the utility function. The objective function of the individual would then be

$$\int_0^\infty e^{-bt}\left(\log C_t + y\ \log\left(1-L_t\right)\right)dt\ . \tag{13}$$

FOC's for the individual w.r.t. *C*, *M*, *B*, are exactly as in the previous problem, matching (2)-(4) above. This allows us again to derive the conclusion that the any equilibrium will have $V \equiv \bar{V}$ (again invoking a backstop fiscal policy and transversality). The individual's FOC w.r.t. *L*, needed here because now *L* is a decision variable for the individual, is

$$\P L: \qquad \frac{y}{1-L} = l\,\frac{W}{P}\ . \tag{14}$$

The firm has two constraints, the budget constraint (8*) (*'d equation numbers are on the original problem handout) and the demand curve constraint defined by (9*)-(10*). We give these the Lagrange multipliers *m* and and *n*, respectively. The firm FOC's are then

$$\P p: \qquad P = m \tag{15}$$

$$\P L: \qquad m\cdot\left(aL^{a-1} - \frac{W}{P}\right) = n\,\frac{aL^{a-1}}{\bar{L}^a} \tag{16}$$

$$\P P: \qquad p = -\frac{d}{dt}\left(2mf\dot{P}\right) + rmf\dot{P} - \frac{WL}{P^2}m + n\,\frac{qP^{-q-1}}{\bar{P}^{-q}}\ . \tag{17}$$

We can already see here from (16) that wage will not always be equal to marginal product. This would only occur if the right-hand side of that equation were zero, which is possible only with $n = 0$, i.e. if the demand curve constraint is not binding. Note that if the demand curve is interpreted as an inequality constraint, so that the firm has the option of not producing, not hiring, and earning zero profits, then $n \geq 0$ is a necessary condition for an optimum. That is, the real wage must always be below the marginal product of labor. This helps motivate the idea that firms still supply the output demanded during booms, despite high wages: they still increase profits with each additional sale.

We must be careful not to choose parameters so that the steady-state of the model implies $n < 0$. It is a matter of interpretation whether, when the steady state has $n > 0$, we need to restrict dynamic paths of the economy to keep $n \geq 0$ at all dates. If firms have the option of shutting down temporarily whenever $n < 0$, they will certainly do so. But we have not really explained how a firm arrives at its downward-sloping demand curve. If distinctions among products are purely technological, and firms own the right to produce a product with a particular technical specification, then they would optimally shut down when $n < 0$. But one can also imagine that firms simply occupy positions in product space (e.g. a physical location, or a spot as the supplier of a trademarked brand) which they lose unless they maintain continuous supply. In this latter case there would still be a requirement that the present value of profits remain positive.

Now we can eliminate the Lagrange multipliers, expand the derivative term in (17), and use the equilibrium relationships that tell us V is constant at $\bar{V}$, $P = \bar{P}$, and $L = \bar{L}$.[1] This results in

$$n = PL^a - \frac{WL}{a} \tag{18}$$

from substituting P for m in (16). The social resource constraint, obtained from the government, firm and consumer budget constraints, is then

$$C \cdot (1 + gV) = L^a - f\dot{P}^2 \ . \tag{19}$$

Using (14), the definition of velocity V, and (2), we can derive

$$W = \frac{\overline{VM} \cdot (1 + 2g\bar{V})y}{1 - L} \ . \tag{20}$$

From (19) and the definition of velocity we get

$$P \cdot \left(L^a - f\dot{P}^2\right) = \overline{VM} \cdot (1 + g\bar{V}) \ . \tag{21}$$

And using (15), (18), the firm budget constraint (the definition of p), and the fact from (5) and (6) that $r \equiv b$ to make substitutions in and rearrange (17), we get

$$2f\dot{P}^2 + 2fP\ddot{P} - 2fbP\dot{P} + \frac{WL}{P} = -p + \frac{nq}{P} = -(1-q)L^a + \frac{WL \cdot (-q/a + 1)}{P} + f\dot{P}^2 \tag{22}$$

We can see now that (20) will let us eliminate W from (21)-(22), producing

$$2f\dot{P}^2 + 2fP\ddot{P} - 2fbP\dot{P} = -(1-q)L^a + \frac{-q}{a}\frac{\overline{VM} \cdot (1 + 2g\bar{V})yL}{P \cdot (1 - L)} \ . \tag{23}$$

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[1] The problem used bad notation. Its demand equation, and some of the FOC's above, use symbols like $\bar{L}$ and $\bar{P}$ to mean average per capita P and L. These "barred" variables change over time. But the problem also uses $\bar{M}$ to refer to the constant at which per capita money holdings are pegged, and we have used $\bar{V}$ in class to refer to constant level of velocity consistent with steady state. For P and L, we will therefore use P^* and L^* to refer to steady-state values, while for all other variables the bar means steady-state value.

If we linearize around deterministic steady state, the $\dot{P}^2$ terms in (21)-(23) disappear (since they are of second order for small deviations from steady state). Linearizing (23) produces

$$2fP^*\ddot{P} - 2fbP^*\dot{P} = -a\cdot(1-q)\left(L^*\right)^{a-1}dL + \frac{-q}{a}\frac{\overline{VM}\cdot\left(1+2g\overline{V}\right)yL^*}{P^*\cdot\left(1-L^*\right)}\left(\frac{dL}{L^*\cdot\left(1-L^*\right)} - \frac{dP}{P^*}\right). \quad (24)$$

Using the linearized version of (21),

$$\frac{dP}{P^*} = -a\,\frac{dL}{L^*} \quad (25)$$

in (24) produces

$$2fP^*\ddot{P} - 2fbP^*\dot{P} = \left((1-q)\left(L^*\right)^a + \frac{q}{a}\frac{\overline{VM}\cdot\left(1+2g\overline{V}\right)yL^*}{P^*\cdot\left(1-L^*\right)}\left(\frac{1}{a\left(1-L^*\right)}+1\right)\right)\frac{dP}{P^*} \quad (26)$$

Completing the answer requires assuming some reasonable parameter values (say $q=2$, $b=.05$, $g=.01$, $a=.7$, $m=1$, $f=1$) and finding the steady state. We already know that $\overline{V}=\sqrt{b/g}=\sqrt{5}$. From (20) and (21) (in its steady-state version with $\dot{P}=0$) we can get

$$\frac{\overline{W}L^*}{P^*} = \left(L^*\right)^a \frac{yL^*}{1-L^*}\frac{1+2g\overline{V}}{1+g\overline{V}} \quad . \quad (27)$$

From (22) we get

$$\frac{\overline{W}L^*}{P^*} = \frac{q-1}{q}a\left(L^*\right)^a \quad . \quad (28)$$

Solving these last two equations gives us

$$\frac{L^*}{1-L^*} = \frac{a}{y}\cdot\frac{q-1}{q}\cdot\frac{1+g\overline{V}}{1+2g\overline{V}} \quad , \quad (29)$$

which is easily solved for L^*. With the suggested parameters the result is $L^*=.343$. Then (20) lets us find $\overline{W}=3.55$ for the initial $\overline{M}=1$ steady state. We can then get from (21) $P^*=4.84$. The linearized differential equation (26) then becomes

$$9.68\ddot{P} - .484\dot{P} - .373dP = 0$$

with characteristic values -.173 and .223. Suppressing the unstable root leaves us with the first-order system

$$\dot{P} + .173dP = 0 \quad . \quad (30)$$

Thus the rate at which P, and all other variables in the linearized system, return to steady state after a perturbation is .173.

If we took the original problem statement seriously, the doubling of $\overline{M}$ would require that L^a double, so that the new initial L became .922. While this is within the feasible range (0,1), it implies a very large initial wage increase that makes the initial n negative and large in absolute value. It therefore would make sense instead to consider a much smaller disturbance in M, say an increase from 1 to 1.1. This requires an increase of only about 12 % in L. The steady state for all real variables is unchanged by the M shift, but the new steady-state W and P are higher in proportion to the increase in M. We can see from (20) that W must initially increase more than in proportion to the increase in M, so it declines toward its new steady state as P rises toward its new steady state. The higher initial real wage is required to make workers agree to supply the higher L. Note that though this model is in some sense "Keynesian", it is unlike simple textbook ISLM models in that it implies that demand-driven expansions make the real wage strongly pro-cyclical, not counter-cyclical. Firms allow the real wage to rise temporarily, reducing their profit margins, while they adjust their prices upward.