Econ. 511b Spring 1997 C. Sims

Suggested Solutions to 2/3 term exam

1a. With $\dot{K} = I$ substituted into (2) and l_t the (current value) Lagrange multiplier on (2), the FOC's to the individual's maximization problem are (round bracket equation references are on the exam, square brackets references are in this answer sheet):

(C): $$1/C_t = l_t \quad [1]$$

(K): $$e^{-bt} l_t r_t Q_t = \frac{d}{dt}\left[e^{-bt} l_t Q_t\right]$$

$$\Leftrightarrow r_t = b - \frac{\dot{l}_t}{l_t} - \frac{\dot{Q}_t}{Q_t} \quad [2]$$

The firm maximizes (3) s.t. (4) at each instant. Let y be the Lagrange multiplier associated with (4) and define $Y = AK^a$. Then the FOC's are:

(C): $$1 = y\,(1/s)\left(qC^s + (1-q)(I+dK)^s\right)^{\frac{1}{s}-1} q\,s\,C^{s-1} = y\,qY^{1-s}C^{s-1} \quad [3]$$

(I): $$Q = y\,(1-q)Y^{1-s}(I+dK)^{s-1} \quad [4]$$

(K): $$-Qr + y\left[aAK^{a-1} - (1-q)Y^{1-s}(I+dK)^{s-1}d\right] = 0$$

$$\overset{[3]}{\Leftrightarrow} y\,aAK^{a-1} = Q(r+d) \quad [5]$$

1b. [4]/[3] implies:

$$Q = \frac{1-q}{q} C^{1-s}(I+dK)^{s-1} \quad [6]$$

We can get rid of C by rewritting (4) as

$$C = \left(q^{-1}(AK^a)^s + q^{-1}(1-q)(I+dK)^s\right)^{\frac{1}{s}} \quad [7]$$

and inserting this into [6]. The resulting expression is:

$$Q = \frac{1-q}{q}\left[q^{-1}(AK^a)^s(I+dK)^{-s} + q^{-1}(1-q)\right]^{\frac{1-s}{s}} \quad [8]$$

For given *K*, this implies a negative relation between *Q*, the price of investment goods, and investment *I*, if $s > 1$.

1c. What was expected here is that you would apply to this model a point made in lecture. In Q-theory models of investment generally the investment decision appears to become "myopic", with current *I* determined by the current *Q*. But when there are adjustment costs, which at the aggregate level is equivalent to when there is a variable relative price of capital and consumption goods, *Q* itself must be set by markets in a forward-looking way. *Q* can generally be expressed as a discounted present value of future returns to capital. Thus an *I* that depends on current *Q* also depends on expectations of future profitability.

To apply this idea to this model we rewrite [2], using [1], as

$$r + \frac{\dot{Q}}{Q} = \frac{\dot{C}}{C} + b \quad . \qquad [9]$$

The flow of direct returns on investment in *K* is *rQ*, so to convert [9] to a relation between current price of *K* and returns to *K*, we write

$$\dot{Q} = \left(b + \frac{\dot{C}}{C} \right) Q - Qr \quad . \qquad [10]$$

Now $b + \dot{Q}/Q$ is the real interest rate, since if we added to the model a zero-net-supply real bond, its interest rate i would satisfy the familiar (under log utility) equilibrium condition $\dot{C}/C = i - b$. If we represent this term by i and the time-t discount factor by

$$\Phi_t = e^{\int_0^{\infty} i_s ds}, \qquad [11]$$

then the only stable solution to [10] will be its forward solution

$$Q_t = \int_0^{\infty} \frac{\Phi_{t+s}}{\Phi_t} Q_{t+s} r_{t+s} ds \quad . \qquad [12]$$

Since by looking at the FOC's for the firm we know that future *Qr* depends in a non-trivial way on future production decisions and future values of *A*, [12] shows that *Q*, and hence *I* , depends on beliefs about the future.

This argument is only heuristic. The argument is correct without having made any reference to s, the production substitution parameter between *C* and *I*. Yet we know that with $s = 1$, so that *C* and *I* are perfect substitutes, we can derive from the firm FOC's, in particular [6], the conclusion that *Q* is a constant, and in fact we know that the existence of an investment function, and its taking a forward-looking form, depends on our having $s > 1$. So an ideal answer would have displayed [12], but then remarked that for it to have the desired implication requires that *Qr* have non-trivial variation independent of Φ, which fails to be true in the case of $s = 1$.

A more precise answer could have been obtained in principle by reducing the system to a single equation in $\ddot{K}$, $\dot{K}$ and *K*, then noting that the linearized equation has one unstable root, which when eliminated leaves an equation relating $\dot{K} = I$ to future disturbances. However, the algebra required to do this in this model was too much to expect you to complete in a 90 minute exam.

This question should have given you more guidance as to how to get started, as it was easy to start down this latter essentially correct route to an answer, but not have time to complete it. Answers that proceed in this correct direction without finishing will get at least some credit.

2a. When $q<0$ transaction costs are negative: $\frac{V^q}{q}<0$ and are unbounded below: $\frac{V^q}{q}\to-\infty$ as $V\downarrow 0$. Given the economy's social resource constraint (from (6) and (7)):

SRC: $$C(1+g\,\frac{V^q}{q})=Y$$

this implies negative consumption for a range of small V's. Moreover, when

$$V\downarrow\left(\frac{-q}{g}\right)^{1/q}:=\tilde{V}\ \ (\tilde{V}\text{ satisfies }(1+g\,\frac{V^q}{q})=0),\ \ C\to\infty\ \text{ for given } Y.$$

2b. The FOC's are (using some of the same intermediate steps as in class):

(C): $$C^{-q}=l\,(1+g\,(q^{-1}+1)V^q)\qquad [13]$$

(M): $$gV^{q+1}=b-\frac{\dot{l}}{l}+\frac{\dot{P}}{P}\qquad [14]$$

(B): $$r=b-\frac{\dot{l}}{l}+\frac{\dot{P}}{P}\overset{[10]}{=}gV^{q+1}\qquad [15]$$

2c. A demand for money relation can be derived from [15]:

$$r=g\left(\frac{PC}{M}\right)^{q+1}$$

Taking logs

$$\log r=\log g+(q+1)(\log P+\log C-\log M)$$

and differentials (keeping $d\log P=d\log C=0$) we obtain for the interest elasticity of money demand:

$$\frac{\P\log M}{\P\log r}=\frac{-1}{1+q}\in(-1,0)\text{ for } q>0$$

2d. With $q = 1$ [13] implies

$$-\frac{\dot{C}}{C} = \frac{\dot{l}}{l} + \left(\frac{g\,(q^{-1}+1)qV^{q-1}\dot{V}}{1+g\,(q^{-1}+1)V^{q}}\right) \qquad [16]$$

With $M = \overline{M}, \frac{\dot{V}}{V} = \frac{\dot{C}}{C} + \frac{\dot{P}}{P}$. Using this with [16] and [14] we get a differential equation for V:

$$\frac{\dot{V}}{V} = \left(\frac{1+g\,(q^{-1}+1)V^{q}}{1+g\,(q^{-1}+2+q)V^{q}}\right)(-b + gV^{q+1}) =: g(V) \qquad [17]$$

This differential equation has a unique steady state $\overline{V}$:

$$\overline{V} = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{q+1}}$$

Since $g(V) > 0$ for $V > \overline{V}$ and $g(V) < 0$ for $V < \overline{V}$, [17] is unstable and, as it is also continuous,

(i) If $V > \overline{V}$, $V_t \rightarrow \infty$

(ii) If $V < \overline{V}$, $V_t \downarrow 0$

Assuming for the moment that we can rule out cases (i) and (ii), the price level is uniquely determined from:

$$\frac{PC}{\overline{M}} = \frac{PY}{\left(1+\gamma\frac{\overline{V}^{q}}{q}\right)\overline{M}} = \overline{V}$$

That is,

$$P = \frac{\overline{V}\left(1+\gamma\frac{\overline{V}^{q}}{q}\right)\overline{M}}{Y}$$

To rule out case (i), note that the SRC implies that, as V goes to infinity, C goes to zero. Combined with the definition of V, this implies that P must go to infinity for velocity to explode. But this is inconsistent with the backstop fiscal policy, which guarantees that the government is always willing to exchange bonds or money for goods at a (upper limit) price level, P^*, so that P cannot exceed P^* on an equilibrium path.

To rule out case (ii), note that the SRC implies that, as velocity goes to zero, C approaches Y, so that P must go to zero as well as V. This implies that real wealth goes to infinity, because the passive fiscal policy keeps real bonds stable (in any case we usually assume $B \geq 0$), and because $M = \overline{M}$. This violates transversality of the private sector. Because agents, taking the price level as given, perceive that they can be better off by consuming part of their ever increasing money balances, this cannot be an equilibrium. A more precise argument is the following.

Suppose that an individual consumes a fraction *d* of money balances from time *t* to *t+Dt*, keeping money balances at *(1-d)M* after *t+Dt*. The new level of consumption from *t* to *t+Dt* is given by:

$$C^*_{t+s}\left(1+gq^{-1}(P_{t+s}C^*_{t+s} \,/\, M^*_{t+s})^q\right)= C_{t+s}\left(1+gq^{-1}(P_{t+s}C_{t+s} \,/\, \overline{M})^q\right)+\frac{d\,\overline{M} \,/\, \Delta t}{P_{t+s}}, \quad \text{for } 0 \le s < \Delta t$$

where *'ed variables refer to the new policy. (Note that *P* will not change if one atomistic individual changes its policy; individuals are price-takers.) Because $M^*_{t+s} \ge (1-d)\overline{M}$ it is easy to check that, as $\frac{\overline{M}}{P_{t+s}} \to \infty,\ C^*_{t+s} \to \infty$. Hence, with log-utility the increase in utility over time *t* to *t+Dt* grows unbounded as $t \to \infty$. (It is appropriate to ignore discounting from time 0 to *t* in this analysis, as all ultility gains and losses occur after time *t*.)

After time *t+Dt* velocity is higher than under the original policy. This lowers consumption from that time onward.

$$C^*_{t+s} = Y\Big/\left(1+gq^{-1}(P_{t+s}C^*_{t+s} \,/\, (1-d)\overline{M})^q\right) < C_{t+s}, \quad \text{for } s > \Delta t$$

However, as *P* goes to zero *C* approaches *Y* under both policies, so that the loss in utility from time *t+Dt* onward goes to zero as $t \to \infty$.

The conclusion is that the proposed 'eating up of real balances' would increase the individual's utility. Hence, agents will not willingly hold $\overline{M}$ along paths with velocity shrinking to zero. Thus, we have ruled out case (ii) as an equilibrium.