Econ. 511b Spring 1997 C. Sims

The Simple Analytics of Price Determination

The logic of price determination through fiscal policy may be best appreciated in an extremely lean model. We include no stochastic elements, we assume there is no money, and we assume that there is no capital accumulation. There is, however, nominal government debt, which individuals perceive as giving them an option of shifting consumption through time.

We assume a representative agent who maximizes

$$\int_0^\infty \frac{C_t^{1-g}}{1-g} e^{-bt} dt \tag{1}$$

with respect to C and B subject to

$$C + \frac{\dot{B}}{P} + t = \frac{rB}{P} + Y \tag{2}$$

$$B \geq 0 \ . \tag{3}$$

Equation (2) is the usual budget constraint, equating consumption, asset accumulation, and taxes to the yield on wealth and exogenous non-asset income Y. Note that it is not a “jump” equation: the time-derivative appearing in it is both a left and right derivative. B can change only over time, via a gap between income and expenditure. The inequality (3) requires that individuals not borrow from the government. Restrictions weaker than (3) would also work, but some such condition preventing individuals from financing arbitrarily large C by rolling over debt (negative B) forever is required.

The government’s instantaneous budget constraint is

$$\dot{B} = rB - Pt \ . \tag{4}$$

The government can be thought of as choosing r, B and t subject to (4), with P taken as given, or it can be thought of as choosing all variables in the system subject to (4), (2) and private optimizing behavior. To close the model we need two more equations characterizing government policy. For example, one of these equations can be a tax-setting, or fiscal policy, equation, while another is an interest-rate-setting, or “monetary” policy equation.

Equations (2) and (4) imply the social resource constraint, which is simply

$$C = Y \ . \tag{5}$$

We define the real interest rate as

$$r = r - \frac{\dot{P}}{P} \ , \tag{6}$$

where the derivative in (6) is a right-derivative, referring to expected inflation from now on.

I. The Private Agent's Problem

FOC's for the private agent are

$$¶C: \quad C^{-g} = l \quad . \tag{7}$$

$$¶B: \quad -\frac{\dot{l}}{P} + \frac{l}{P}\frac{\dot{P}}{P} + b\,\frac{l}{P} = r\,\frac{l}{P} \quad . \tag{8}$$

Substituting and manipulating the result, we obtain

$$g\,\frac{\dot{C}}{C} = r - \frac{\dot{P}}{P} - b \quad . \tag{9}$$

Equation (9), having been derived from Euler equations, is a jump equation. The derivatives in it are forward-looking, right-derivatives.

In the first examples below we will maintain the assumption that *Y*, and hence in equilibrium (via (5)) *C*, is constant, which implies via (9) and (6) that $r = b$.

II. Pegging the Real Primary Surplus and the Nominal Interest Rate

Suppose policy fixes the primary surplus as a constant $t = \bar{t}$ and the nominal interest rate as a constant $r = \bar{r}$. Because in equilibrium under our assumptions $r = b$, we can write

$$\bar{r} = b + \frac{\dot{P}}{P} . \tag{10}$$

This means that

$$P_t = P_0 e^{(\bar{r}-b)t} \quad , \tag{11}$$

for some initial P_0. We can rewrite the government budget constraint (4) as

$$\dot{B} = \bar{r}B - \bar{t}P_0 e^{(\bar{r}-b)t} \quad . \tag{12}$$

This is an unstable linear difference equation, whose general solution is

$$B_t = \frac{P_0 e^{(\bar{r}-b)}\bar{t}}{b} + k e^{\bar{r}t} \quad , \tag{13}$$

where *k* is some constant. Dividing through by *P*, we get

$$\frac{B}{P} = \frac{\bar{t}}{b} + \frac{k e^{bt}}{P_0} \quad . \tag{14}$$

Thus the only value for P at $t = 0$ that is consistent with real government debt not exploding exponentially at the rate b is the P_0 that satisfies (14) at $t = 0$ with $k = 0$, i.e.

$$\frac{B_0}{P_0} = \frac{\bar{t}}{b} \quad . \tag{15}$$

In words, the initial price level must adjust to make the real value of the outstanding nominal debt equal to the discounted present value of future net surpluses.

But how do we know that B/P could not explode exponentially in equilibrium? We can rule out $k<0$ by the fact that, from (14), we can see that this implies that B would eventually become negative, violating the no-borrowing constraint on individual behavior. Individuals would therefore see themselves as not having enough resources to finance the $C=Y$ consumption level and also pay their taxes. This would generate an attempt to save, driving down the price level. If $k>0$, real debt grows without bound. An individual who is accumulating wealth at this steady rate b will see him or herself as able to increase consumption at all dates by, say, $\frac{bk}{2P_0}$. The individual, using his or her budget constraint to calculate the effects of this increase in C, would conclude that debt would no longer follow (12), which is an implication of the individual's budget constraint under $C=Y$, but instead would follow

$$\dot{B} = \bar{r}B - \bar{t}P_0 e^{(\bar{r}-b)t} + P\cdot\left(Y-C\right) = \bar{r}B - \left(P_0 e^{(\bar{r}-b)t}\right)\left(-\frac{bk}{2P_0} - \bar{t}\right). \tag{16}$$

This is another unstable linear difference equation. Its general solution is, analogous to (14),

$$\frac{B}{P} = \frac{\bar{t}}{b} + \frac{k}{2P_0} + \frac{xe^{bt}}{P_0} \ . \tag{17}$$

Because this equation, for a potential time path of an individual's debt perceived by the individual as possible, must satisfy the same initial condition as the equilibrium for the economy, it must imply the same B_0/P_0. Equating (14) and (17) at $t=0$ implies $x=k/2>0$, i.e. that wealth in the form of government debt still grows exponentially forever, despite the higher consumption. It is easy to see that this would be true so long as the amount of the increase in the constant level of consumption remains no higher than k/P_0.

We can conclude, therefore, that only the solution in which the initial price level satisfies (15) is an equilibrium: If P_0 is lower than that, so initial wealth is higher, the FOC's and constraints imply that wealth will accumulate forever, so rapidly that individuals would perceive the possibility of financing a permanent increase in C out of their wealth; and if P_0 is higher than that, so initial wealth is lower, the FOC's and constraints imply that individual wealth will become negative in finite time, violating the solvency constraint on individuals (3).

Note that we did not arrive at the unique equilibrium here simply by "ruling out explosive solutions". There is an explosive component to the correct solution. Prices rise or shrink (depending on whether $\bar{r}>b$ or $\bar{r}<b$) exponentially in equilibrium, according to (11), and nominal debt B must rise or shrink in proportion to P.

Note also that, though the initial debt level B_0 can be any positive number, it cannot be zero. This is because the budget constraint (12) then implies initial $\dot{B}<0$, which cannot occur when individuals have the solvency constraint (3).

Equilibrium in this economy displays a kind of "quantity theory of debt" determination of the price level. The higher the initial level of *B*, the higher the initial *P*, and *B* and *P* move in proportion to one another. In fact, if we set $\bar{r} = 0$, we get an ordinary "quantity theory of money". With zero nominal interest rate, the "debt" in this model becomes in effect non-interest-bearing money. Despite the absence of a transactions motive for holding money, people hold it because there is steady deflation at the rate $\bar{r} - b = -b$, making money an asset with an attractive return. The government maintains the deflation by taxing away a fixed proportion of the nominal stock of money per unit time. This causes prices to drop in parallel with the decline in *B* (which in this case we may as well refer to as *M*), leaving real money balances M/P constant.

III. Indeterminacy from Passive Fiscal Policy

Economists sometimes have written as if it were clear that when governments issue new debt they are automatically committing themselves to the future taxes that will be needed to pay off the debt and interest on it. This is not true – governments can issue debt without any commitment to increased future taxes, and the result will be price inflation. But suppose a government is persuaded that it should commit to future taxes when it increases debt. One policy that has this effect sets the real primary surplus t to respond positively to the level of outstanding real debt, for example

$$t = f_0 + f_1 \frac{B}{P} \quad . \tag{18}$$

In order that (18) represent to paying the interest on the real debt, we require $f_1 > b$, i.e. the primary surplus increases by more than the increase in real interest rate payments as B/P increases. It is easy to see that substituting this into the government budget constraint (4) produces

$$\dot{B} = (r - f_1)B - Pf_0 \quad . \tag{19}$$

If we assume as before that interest rate policy simply pegs $r = \bar{r}$ at all times, we have the same class of solutions (11) for *P* and (19) becomes a constant-coefficient linear differential equation, with general solution

$$B = e^{(\bar{r}-b)t} \frac{-f_0 P_0}{f_1 - b} + ke^{(\bar{r}-f_1)t} \quad \text{, or} \tag{20}$$

$$\frac{B}{P} = \frac{-f_0}{f_1 - b} + \frac{ke^{(b-f_1)t}}{P_0} \quad .. \tag{21}$$

Note that, under our assumption that $f_1 > b$, and assuming also $f_0 < 0$, (21) describes a stable time path for real debt, regardless of the initial price level. No initial price level can be ruled out as too low, as this condition does not imply ever-growing wealth for individuals, and none as too high because this condition does not imply that debt must become negative. The initial price level is indeterminate.

In Leeper's local linear analysis of an economy with money, this kind of fiscal policy is called passive, and he concludes that to guarantee uniqueness, it must be paired with an active interest rate policy, that is, one that aggressively increases the interest rate in response to inflation. A simple example of an active interest rate policy is one that sets

$$r = q_0 + q_1 p \,, \tag{22}$$

where $p = \log P$. Regardless of policy, the government budget constraint can be rewritten in real terms to produce

$$\dot{b} = rb - t \quad , \tag{23}$$

where $b = B/P$. Note that if we replace (4) with (23) we lose some information, formally. Equation (4) is a non-jump equation, while (23), because it uses the forward-looking relation between the real and nominal interest rate, is a jump equation. However, (23) is a valid equation, and we can use it provided we keep in mind that it is *B*, not *b*, that is given by history. Substituting (18) into (23) produces

$$\dot{b} = \left(r - f_1\right)b - f_0 \quad . \tag{24}$$

With $f_1 > b = r$, (24) is a stable differential equation. Now using the interest policy equation (22) and the real interest rate definition (6) to eliminate *r*, we arrive at

$$q_0 + q_1 p = b + \dot{p} \quad . \tag{25}$$

This is an unstable equation in *p*. It has the unique stable solution $p = (b - q_0)/q_1$. But the unstable paths for *p* generate no violations of equilibrium conditions. The real rate of return remains fixed at $r = b$, even as the nominal rate diverges to $\pm\infty$. The real value of the debt converges exponentially toward its positive steady state value of $-f_0/\left(f_1 - b\right)$, regardless of where it starts from, even though nominal debt explodes at ever-increasing exponential rates.

Thus an active interest rate policy succeeds only in making it very likely that the price level will explode. It does not eliminate the indeterminacy that arises because (24) is a stable equation. The reader may wish to confirm that with the $t = \bar{t}$ policy of the previous section, an active interest rate policy like (22) leaves the initial price level determinate, while implying in general that the price level and nominal debt follow (more than exponentially) explosive paths.

IV. A Balanced Budget Amendment

In the US, the possibility of a policy which makes the conventional deficit constant is being discussed. In our notation this would be a policy of $\dot{B} = 0$. This implies in turn

$$rB = Pt \quad . \tag{26}$$

Usually this policy is taken to be a prescription for taxes and expenditures, not a prescription that nominal interest rates on debt be kept low enough to maintain a zero deficit. So suppose we have an active interest rate policy like (22) to go with the balanced budget. As we have already observed, (22) implies that there is a unique initial price level consistent with non-explosiveness of the price level. This will imply a unique initial *r* from (22), and then from (26) and the given initial value of *B* a unique initial tax level *t*. Explosively decreasing *p* implies, because *B* is fixed, explosively increasing B/P. As we have discussed before, this is inconsistent with individual optimization. But explosively increasing p implies only explosively decreasing toward zero B/P. (The fact that it is log p that decreases exponentially, becoming negative, means that P itself only shrinks toward zero, remaining positive.) This does not violate individual optimality conditions and it leads to no violation of solvency constraints. Thus a conventional balanced budget policy, even coupled with active interest rate policy, leads to indeterminacy of the price level. It rules out low prices followed by explosive deflation, but not high initial prices followed by explosive inflation.

We could, however, couple (26) with an active fiscal policy like $t = \bar{t}$. This converts (26) to a prescription for setting interest rates to balance the budget. Using the definition of the real rate and its constancy in equilibrium, we find that (26) becomes

$$\dot{p} + b = \frac{P\bar{t}}{B} \ . \tag{27}$$

While not a linear differential equation, this is an unstable equation in P with a unique constant solution: $P = \dfrac{Bb}{t}$. We might be tempted to conclude that (27) will once again generate upwardly explosive P paths when initial P is too high that are nonetheless consistent with equilibrium. However, this nonlinear equation can be solved analytically to imply some interesting behavior. Rewriting it, we can find

$$\frac{\dot{P}}{P \cdot \left(P\dfrac{t}{B} - b \right)} = 1 \ . \tag{28}$$

We can expand the left-hand side in partial fractions and integrate, obtaining (when $Pt/B > b$)

$$-\frac{1}{b}\log P + \frac{1}{b}\log\left(P\frac{t}{B} - b \right) = t + k \ , \tag{29}$$

where k is a constant of integration. Exponentiating both sides, we arrive at

$$\frac{P\dfrac{t}{B} - b}{P} = e^{(k+t)b} \ . \tag{30}$$

Since its left-hand side is bounded above by t/b for positive P, (30) implies that from any initial value of P, we reach infinite P in finite time. That is, when the initial P is above its steady-state value, inflation must be so explosive that the price level goes to infinity in finite time.

How do we interpret this result? The problem is that to characterize behavior properly we need to consider policy behavior after the date at which P becomes infinite. But (26) becomes a nonsense equation with P infinite. If we interpret $P = \infty$ as implying that there is no debt on which to pay interest, then the commitment to $t = \bar{t}$ is incompatible with a balanced budget after P becomes infinite. One has to complete the model with a specification for policy in this post-hyperinflation period. If we simply say that t drops back to zero when P reaches infinity, then there is indeterminacy – essentially the initial real value of the debt can be larger or smaller depending on how long there is before P reaches infinity, and hence how long there is for the

$t = \bar{t}$ policy to be sustained. If we suppose instead that the government will drop the balanced budget policy after *P* reaches infinity, maintaining a conventional surplus and $t = \bar{t}$ even after the real debt has evaporated, private agents will see themselves as needing to borrow at this point to pay taxes. Anticipating this, they will feel insolvent earlier, save more, and thereby reduce the price level to that consistent with non-explosive equilibrium.

V. The Ill-Fated Exercise

Now we have the foundation to answer the exercise you were asked to complete last week. We have in fact already answered, or shown to be unanswerable, some of the questions. Question 3 on the exercise concerned the balanced-budget model of the last section, with fixed primary surplus. As we have seen, the question can't be answered without considering how to respecify policy after the price level reaches infinity. However, if you relied on the plausible (but not justifiable from what you were given in the problem alone) assumption that solutions with explosive real debt were not equilibria, you would have concluded that there is a unique initial price level. The same analysis applies to question 4. Though we do not discuss it in section IV, the passive fiscal policy of question 4 leads to no fundamental change in the analysis if $f_0 > 0$. In that case there is still the problem that the price level goes to infinity in finite time if it starts too high, making the commitment to positive primary surplus and zero conventional deficit contradictory. If the constant term f_0 in the fiscal rule is made negative, however, and $f_1 > b$, the analogue of(27) becomes a stable equation in *P*, so there is non-uniqueness.

Question 5 couples an active interest rate policy with an active fiscal policy and asks that you prove non-existence. But as we saw in section II, active fiscal policy does generally deliver a unique initial price level in this model, with interest rate policy simply determining the time path of prices after the initial date. However, the exercise did not use these notes' version of an active interest rate policy, equation (22), but instead an equation that replaced *p* in (22) with the price level *P*. With such an interest rate policy equation, *P* is implied to reach infinity in finite time, again producing an ambiguity about what the policy equation for interest rates means when it implies infinite interest rates.

Question 1 was straightforward. Solving the government budget constraint forward implied

$$b = e^{-.1t} \frac{\bar{t}}{.1 + b} \ . \tag{31}$$

We know we can do this because (23) implies that for any other solution path for debt, either real debt explodes, violating individual optimization, or eventually becomes negative, violating solvency. Since B_0 is given, we can use (31) at $t = 0$ to determine a unique P_0. We also know from the private sector FOC's that

$$\dot{p} + b = \bar{r} \tag{32}$$

and therefore that

$$P = P_0 e^{(\bar{r} - b)t} \ . \tag{33}$$

Since we know the unique initial P_0, (33) gives us the full time path of *P*, and then (31) implies the full time path of *B* and $\dot{B}$. Using the numerical values supplied in the question, I concluded $P = 3e^{.05t}$, $B = 2e^{-.05t}$, $\dot{B} = -.1e^{-.05t}$, $b = \frac{2}{3}e^{-.1t}$.

Question 2, as already pointed out in class, was internally inconsistent. Because of the declining *Y* and hence *C*, the real interest rate is implied to be negative. Consumers must hold non-negative government debt, and they have a stream of income that is dwindling exponentially to zero. The government debt they hold pays a negative return. Thus they do not have enough resources to pay a fixed real government tax forever. The fixed-$\bar{\tau}$ policy proposed in the question is unsustainable.

Question 6 is a combination of passive fiscal policy with active interest rate policy, with the level rather than log of price entering the interest rate rule. This form of interest rate rule leads to

$$\dot{p} + b = q_0 + q_1 P \quad . \tag{34}$$

This equation has the same form as (27) and leads to the same conclusion, that high initial prices lead to infinite prices in finite time. However here, because of the passive fiscal rule, the real value of debt follows a stable time path regardless of what happens to the price level. What happens the instant after the nominal debt and the nominal price level have both reached infinity is even more of a puzzle than in the model with fixed *B* (since with fixed *B* real debt vanishes as the price level reaches infinity). Also, here low initial values of *P*, though they lead to rapidly shrinking *P*, do not lead to explosion of B/P and thus do not violate individual optimization. So the problem, in assuming there was a unique solution, was mistaken.

However, one could have gone ahead and assumed we are interested in the unique stable solution, which exists. It has *P* fixed at $(b - q_0)/q_1$ and *b* following whatever time path is implied by the initial B/P and the equation

$$\dot{b} = (b - f_1)b - f_0 \quad . \tag{35}$$

If we began in steady state, we would have $b = f_0/(b - f_1)$, and thus $B = f_0 \cdot (b - q_0)/(q_1(b - f_1))$. You were asked, though, to suppose that at $t = 0$ q_0 jumps from .03 to .05, while b was assumed to be .5. With this choice of q_0, there is no stable price level. The price level reaches infinity from any finite positive starting value in finite time.

So there is no good answer to question 6.