The General Linear RE Model

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Outline

- The basic idea behind eigenvector decomposition approaches to solving linear RE models
- Canonical forms, continuous and discrete time
- What determines existence and uniqueness
- Allocating effort between yourself and the computer

Our most general canonical form

$$\Gamma_0 y(t) = \Gamma_1 y(t-1) + C + \Psi z(t) + \Pi \eta(t), \qquad t = 1, \ldots, T\,. \quad (1)$$

C: a vector of constants
$z(t)$: an exogenous random disturbance
$\eta(t)$: an expectational error

All we know about $\eta(t)$ is that $E_t \eta(t+1) = 0$, all t. The actual values of $\eta(t)$ have to be determined in solving the model.

Note: No $E_t x(t+1)$ terms in the system. We've replaced any such term by

$$x(t+1) - \big(x(t+1) - E_t x(t+1)\big) = x(t+1) - \eta(t+1)\ .$$

Convention: Anything dated t is known at t, i.e. $E_t x(t) \equiv x(t)$ for any x.

Why a Canonical Form?

- It is some work to get a model into this form. Models often have more than one lag. They often have $z(t)$ and $\eta(t+1)$ in the same equation. They often have $E_t x(t+s)$ terms with $s > 1$. But for this form, the work is modest.

- Once the model is in a canonical form, the solution set can be described automatically, by the computer.

Comparison to Blanchard-Kahn form

$$\begin{bmatrix} y_{t+1} \\ E_t x_{t+1} \end{bmatrix} = A \begin{bmatrix} y_t \\ x_t \end{bmatrix} + z_t \ , \qquad y_0 \text{ given.}$$

Requires identity in place of Γ_0. $\Pi = \begin{bmatrix} 0 \\ I \end{bmatrix}$. That $t+1$ replaces t is irrelevant.

Mapping to DP FOC's canonical form

$$DU_t = -\beta E_t[DV_{t+1} D_C f_{t+1}]$$
$$DV_t = \beta E_t[DV_{t+1} D_S f_{t+1}]$$
$$S_{t+1} = f(S_t, C_t, \varepsilon_{t+1})$$

This is a linear system in the LQ case. First two blocks of equations correspond to an identity block in Π, while last block corresponds to a 0 block in Π. Ψ is zero in positions corresponding to the first two equation blocks. Bottom third of Γ_0 is an identity.

Example

$$y_t = -\theta(r_t - E_t\pi_{t+1}) + E_t y_{t+1} + \varepsilon_t \tag{2}$$

$$\pi_t = \gamma y_t + \beta E_t \pi_{t+1} + \nu_t \tag{3}$$

$$\Gamma_0 = \begin{bmatrix} 1 & \theta \\ 0 & \beta \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix}, \quad \Pi = \underset{2\times 2}{I},$$

$$\Phi = \begin{bmatrix} -1 & 0 & \theta \\ 0 & -1 & 0 \end{bmatrix}.$$

What `gensys.m` Produces

- existence: is there any solution?
- uniqueness: is there at most one solution? (Non-existence and non-uniqueness can coexist.)
- completeness: are there as many equations as variables?

$$y(t) = \Theta_1 y(t-1) + \Theta_c + \Theta_0 z(t) + \Theta_y \sum_{s=1}^{\infty} \Theta_f^{s-1} \Theta_z E_t z(t+s) \quad (4)$$

Θ_1 :	`G1`
Θ_c :	`C`
Θ_0 :	`impact`
Θ_y :	`ywt`
Θ_f :	`fmat`
Θ_z :	`fwt`

Impulse responses

- Impulse responses trace out the effect on the system of unit increases, lasting only one period, in elements of the z vector.

- If z is i.i.d.,and the eigenvalues of Θ_1 are all less than one in absolute value, we can write

$$y(t) = (I - \Theta_1)^{-1}\Theta_c + \sum_{s=0}^{\infty} \Theta_1^s \Theta_0 z(t-s)\ . \tag{5}$$

This is called the **moving average representation** of $y(t)$.

- The elements of the sequence of matrices $\Theta_1^s \Theta_0$ are the impulse responses, with the i'th row and j'th column giving the response of variable y_i to a unit disturbance in disturbance z_j.

- Even when Θ_1 has some eigenvalues on or outside the unit circle, the "impulse response" interpretation remains valid, though (5) does not hold.

- When z is not i.i.d, the impulse responses depend on how expected future z's react to a change in current z, and thus can't be determined without expanding the model to describe explicitly z's serial dependence properties.

Impulse responses, cont.

- Impulse responses are often displayed by plotting the i, j'th element of this impulse response matrix as a function of s.
- Though impulse responses contain no information not available in principle in Θ_0 and Θ_1, they are usually easier to interpret.
- They display "typical modes of behavior" for variables in the system and fit an "if this happens, then that happens" interpretation.
- So they are suggestive of — sometimes too suggestive of — causal interpretations.

The Details, for a Simplified Canonical Form

- $\Gamma_0 = I$

- Stability conditions:

$$E_s\left[\phi_i y(t)\xi_i^{-t}\right] \underset{t\to\infty}{\longrightarrow} 0, \quad i = 1, \ldots, q \tag{6}$$

- Jordan decomposition

$$\Gamma_1 = P\Lambda P^{-1}$$

Λ is "almost diagonal", with "Jordan blocks" down the diagonal.

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_m \end{bmatrix} \tag{7}$$

$$\Lambda_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix} \tag{8}$$

- $w(t) = P^{-1}y(t)$, so

$$w(t) = \Lambda w(t-1) + P^{-1}C + P^{-1}\big(\Psi z(t) + \Pi\eta(t)\big)$$

- Consider block j:

$$w_j(t) = \Lambda_j w_j(t-1) + P^{j\cdot}C + P^{j\cdot}\big(\Psi z(t) + \Pi\eta(t)\big)$$

- Solving backward yields

$$\begin{aligned} w_j(t) = \Lambda_j^t w_j(0) + (I-\Lambda_j)^{-1}(I-\Lambda_j^t)P^{j\cdot}C \\ + \sum_{s=0}^{t-1} \Lambda_j^s P^{j\cdot}(\Psi z(t-s) + \Pi\eta(t-s)) \end{aligned} \quad (9)$$

- If w_j is of length m_j, then the elements of Λ_j^t are products of polynomials in t of order at most m_j with λ_j^t, where λ_j is the diagonal element of Λ_j.

- Therefore if there is any i such that $\phi_i P^{j\cdot} \neq 0$ and $\lambda_j \geq \xi_i$, the only solution for w_j that satisfies the stability conditions is the forward solution

$$w_j(t) = (I - \Lambda_j)^{-1} P^{j\cdot} C - \sum_{s=1}^{\infty} \Lambda_j^{-s} P^{j\cdot} E_t[\Psi z(t+s)]$$

- In the special case where $E_t z(t+1) \equiv 0$, the last term drops and w_j must be a constant. But from (9), $w_j(t)$ has in this case one-step-ahead prediction error (innovation)

$$P^{j\cdot}(\Psi z(t) + \Pi \eta(t)) = 0\,. \tag{10}$$

- For every j whose root needs to be "suppressed", we get such an equation. Stacking up the corresponding $P^{j\cdot}$'s into a matrix P^u (u for “unstable”), we get

$$P^u \Psi z(t) = -P^u \Pi \eta(t) \; . \tag{11}$$

Two kinds of "existence"

- If the space spanned by the columns of $P^u\Pi$ includes all the columns of $P^u\Psi$, then for every possible $z(t)$ we can solve for $\eta(t)$ from (11). Under this condition there is a stochastic process that solves the model. Notice that the model should be formulated so that $z(t)$ can take on arbitrary values.

- The model starts up at time 0, taking $y(-1)$ as given. The solution implies linear restrictions on $y(t)$ that are satisfied for $t = 0, \ldots, \infty$, but $y(-1)$ need not satisfy those restrictions. That is, the solution will imply that the unstable w_j's must take on specific fixed values, while it may be that $w_j(-1)$ differs from its constrained value.

- This means that it is possible for a solution *process* to exist, while no solution that starts up with a particular given $y(-1)$ value exists.

- The condition for existence starting from arbitrary $y(-1)$ is that $P_u \Pi$ be of full row rank. This is obviously stronger than the condition that it span the columns of $P_u \Psi$.

- In most economic models it is natural to take y_{-1} as arbitrary. `gensys` uses the stronger condition in checking for existence, though in older versions it used the weaker condition.

- Economic models where the two conditions give different answers are not common, but they do arise.

Uniqueness

- If the space spanned by the *rows* of $P^u\Pi$ contains all the rows of $P^s\Pi$, where P^s is the matrix formed from all the rows of P^{-1} not contained in P^u, then the value of $P^u\Pi\eta(t)$ determined by (11) also determines the value of $P^s\Pi\eta(t)$, and we have uniqueness.

Root counting

The classic Blanchard-Kahn paper analyzing existence and uniqueness for linear rational expectations systems assumed regularity conditions that allowed invoking the simple condition that the number of unstable roots must match the number of expectational, i.e. forward-looking, equations. In our notation, this is just the requirement that the number of unstable roots must match the number of columns in Π. This criterion usually works, and in any case is a useful rule of thumb.

A simple case of root-counting failure

$$y_t = 2y_{t-1} + \varepsilon_t$$
$$x_t = 2E_t x_{t+1} \; .$$

There is one unstable root — in the y equation — and one expectational equation, the x equation. But the system is "decoupled" — it is really two independent systems. So there is no way that the η_t from the x equation can adjust to keep y_0 at its stable value of 0.

How rare is decoupling?

Note that there are systems close to this one in some sense for which the root-counting gives the right answer. This is an argument made by Onatski, who suggests that this means that decoupled systems are rare. If we alter the y equation to become

$$y_t = 2y_{t+1} - .0001x_{t-1} \, ,$$

The conditions for existence and uniqueness are met. However the solution is weird: the stability condition is $x_t = 15000y_t$, and as we make the system "closer" to the original, by making the coefficient on x_{t-1} in the y equation smaller, the coefficient on y_t in the stability equation and the variance of x_t and η_t blow up. In other words the *solutions* to the system do not get closer in any meaningful sense.

Decoupled systems do turn up in economic models, and in a large system they are generally not as obvious as in this example. Approximately decoupled systems may also turn up, in which conditions for existence and uniqueness are met, but solutions are exotic because of the near-decoupling.