MID-TERM EXAM

(1) (45 minutes) Consider a model in which a representative agent has the objective function

$$\max_{C,K,B} \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \tag{1}$$

and faces the constraints at each period $t = 0, \ldots, \infty$

$$C_t + K_t + \frac{B_t}{P_t} = A_t K_{t-1} + R_{t-1}\frac{B_{t-1}}{P_t} - \tau_t \tag{2}$$

$$B_t \geq 0 \tag{3}$$

$$K_t \geq 0\,. \tag{4}$$

General Comment: Surprisingly many people did not recognize that, with the government budget constraint given as was put on the board at the beginning of the exam,

$$\frac{B_t}{P_t} = R_{t-1}\frac{B_{t-1}}{P_t} - \tau_t\,,$$

there is no real resource use by the government, so taxes and debt have no effect on the real allocation. Taxes are lump sum, prices are flexible, there is a single representative agent, and we are therefore in a world of Ricardian equivalence. Thus variations in interest rate and tax policy affect only inflation and debt; they have no effect, in equilibrium, on the time path of consumption.

(a) Suppose A_t is constant and monetary and fiscal policy fix $R_t = A$ and $\tau_t = \bar{\tau}$. Show that if agents have perfect foresight, there is an equilibrium in which $\kappa = K_t/C_t$ is constant, and find the value of κ as a function of A and β. Display the transversality condition or conditions for this model and verify that they hold for your proposed solution. Assume $A > 1, 0 < \beta < 1, \gamma > 1, \bar{\tau} > 0$.
The FOC's for the agent's problem are

$$\begin{aligned}
\partial C: &\quad C_t^{-\gamma} = \lambda_t \\
\partial K: &\quad \lambda_t = \beta E_t[\lambda_{t+1} A_{t+1}] \\
\partial B: &\quad \frac{\lambda_t}{P_t} = R_t \beta E_t\left[\frac{\lambda_{t+1}}{P_{t+1}}\right].
\end{aligned}$$

Using the government budget constraint, the social resource constraint becomes

$$C_t + K_t = A_t K_{t-1}\,.$$

If $K_t/C_t = \kappa_t$ and $A_t = A$ are constant, the social resource constraint is

$$\kappa^{-1} + 1 = A\frac{K_{t-1}}{K_t}\,.$$

This implies a constant growth rate for K, and hence, because κ is constant, the same growth rate for C. But from the K FOC, we have

$$\left(\frac{C_{t+1}}{C_t}\right)^{\gamma} = \beta A$$

If we define $g = C_{t+1}/C_t = K_{t+1}/K_t$, this last equation determines $g = (\beta A)^{1/\gamma}$. Then from the SRC we can get $\kappa = g/(A-g)$. Note that this equilibrium can exist only if $A > g$, i.e.

$$A^{\gamma-1} > \beta$$

So long as $\beta \in (0,1)$, in the case of log utility ($\gamma = 1$) this is no restriction at all, and for $\gamma \geq 1$ it is satisfied for any $A \geq 1$. For $\gamma \in [0,1)$, it is satisfied for any $A < \beta^{-1}$. For the $\gamma < 1$ cases $g < 1$, so the economy shrinks. When $\gamma > 1$ and A is too small for a constant-κ equilibrium, consumption is pulled toward the present. For the $\gamma < 1$ cases with A too large, utility can be made arbitrarily large by delaying consumption arbitrarily long.

It is easy to check that all the Euler FOC's are satisfied in the constant-κ case. With $R = A$ as we have assumed, the price level is constant. Debt can be constant at $B/P = \tau/(A-1)$.

Because both B and K are constrained to be non-negative, we can treat their TVC's separately, and they are $E[\beta^t\lambda_t K_t] \to 0$ and $E[\beta^t\lambda_t B_t/P_t] \to 0$. We can use these "standard" form FOC's because the state variables B and K are both valuable (have positive shadow prices) at all dates in the solution, both are non-negative, for both 0 is a feasible value for the agent, and neither enters the objective function directly. The TVC for debt is certainly satisfied, as real debt is constant and $\lambda = C^{-\gamma}$ shrinks at the rate $A\beta$, so $\beta^t\lambda_t$ shrinks as A^{-t}. For K, since K grows at the rate $g = (A\beta)^{1/\gamma}$ and $\lambda = C^{-\gamma}$ shrinks at the rate $(A\beta)^{-1}$, the TVC is satisfied iff $\beta(A\beta)^{1/\gamma-1} < 1$. It is easy to check that this condition holds whenever the condition above for the existence of a constant-κ solution holds.

(b) Show that solutions to the Euler equation first order conditions other than the fixed-K/C solution you have considered above violate the transversality conditions or the $B \geq 0$, $K \geq 0$ constraints.

The K and C FOC's together imply that C grows at the constant rate g we calculated above. The SRC divided by C_t is then a difference equation in $\kappa_t = K_t/C_t$, with the coefficient on κ_{t-1} given by $A/g = A^{1-1/\gamma}\beta^{-1/\gamma}$. The K TVC requires that $(\beta(K_t/C_t)C_t^{1-\gamma})^t$ go to zero. But this object, with K/C growing at the rate A/g, grows at the rate $\beta A^{1-1/\gamma}\beta^{-1/\gamma}A^{1/\gamma-1}\beta^{1/\gamma-1} = 1$, and therefore does not converge to zero. So the only solution consistent with the TVC is the constant-κ solution.

The question did not suggest checking for solutions in which B/P is not constant, but the same issue arises there. The FOC's imply that if any debt is held, P must be constant if $R = A$. But then the GBC becomes an unstable difference equation in real debt, with coefficient $R = A$ on lagged real debt. If real debt is not constant, it grows at rate A^t, and this would violate the debt TVC, by the same type of argument we used above for the K TVC.

(c) Suppose now that the economy has been in the constant-K/C equilibrium you derived above, but at time t_0 the value of A unexpectedly shifts downward, while remaining above one. Suppose that, though surprised, everyone understands that A has changed and believes the change is permanent. Determine how K and C behave before, at, and after time t_0.

C and K simply shift immediately to the equilibrium with the new values of κ and g. To determine what happens at t_0, we use

$$C_{t_0} + K_{t_0} = A' K_{t_0-1} ,$$

where A' is the new value of A. The right-hand side is given at t_0, and the left-hand side is divided up to satisfy the new $\kappa' = K_{t_0}/C_{t_0}$. What happens to P and B depends on fiscal policy, as discussed in the next part.

(d) Suppose policy maintains τ fixed at the same value $\bar{\tau}$ before and after t_0, while changing R_t to match the new value of A. What happens to the level of real debt B/P, the price level, and the inflation rate, at and after t_0 in this case?

The real debt must increase immediately to the new, larger $B_{t_0}/P_{t_0} = b_{t_0} = \bar{\tau}/(A'-1)$, where A' is the new value of A. The budget constraint at time t_0 tells us

$$b_{t_0} = \frac{\bar{\tau}}{A'-1} = A b_{t_0-1}\frac{P_{t-1}}{P_t} - \bar{\tau} .$$

From this and the fact that $b_{t_0-1} = \bar{\tau}/(A-1)$ we can derive $P_t/P_{t-1} = A(A'-1)/(A'(A-1))$. This means that, since $A-1$ drops proportionally more than A itself, there is a deflationary jump in the price level at t_0. After that, the price level is constant as before.

(e) Suppose instead that policy keeps both R and τ fixed at the same values before and after t_0 — i.e., R_t remains at the old value of A even after time t_0. What happens to the level of real debt B/P, the price level, and the inflation rate, at and after t_0 in this case?

Since the interest rate was fixed at A at $t_0 - 1$, our calculation of what happens to debt and the price level at time t_0 is unchanged. But the interest rate at t_0 is now too high to make the real return on debt match that on K if there is no inflation. So at dates $t > t_0$, we will have to have $P_{t+1}/P_t > 1$ in order to maintain the real return on debt $RP_t/P_{t+1} = AP_t/P_{t+1} = A' < A$.

(f) Is there a combination of one-time, permanent changes in R and $\bar{\tau}$ that would keep both the price level and the inflation rate constant through time t_0 and beyond? If so, what does the policy imply about the conventional deficit or surplus at and after time t_0? [Recall that τ_t is the real *primary* surplus. The conventional deficit includes "interest expense" and is equal to $B_t - B_{t-1}$, the change in the nominal debt.]

The price level can be kept the same in the two steady states by dropping $\bar{\tau}$ to $\bar{\tau}'$, so that $\bar{\tau}'/(A' - 1) = \bar{\tau}/(A - 1)$. To keep inflation constant after t_0, R has to drop to $R' = A'$. In the new steady state, $\bar{\tau}'$ is exactly enough to cover interest expense and thereby keep both nominal and real debt constant. But at t_0, the interest on the previous period's debt is still at the old $R = A$ rate. If τ has dropped to the new $\bar{\tau}'$ value at t_0, the interest expense will not be covered, nominal debt will increase, and the price level will rise to keep the real debt constant. So to keep the price level constant, the drop from $\tau = \bar{\tau}$ to $\tau = \bar{\tau}'$ must be delayed until $t_0 + 1$.

(2) (45 minutes) In class we derived a "New Keynesian Phillips Curve" under the assumption of Calvo pricing — firms have a chance to change prices at random times and otherwise must keep prices fixed. Here you are asked to derive a New Keynesian Phillips curve with quadratic adjustment costs.

As in the model in the lectures, there is a continuum of monopolistically competitive firms, with each firm i facing a demand curve

$$\frac{q_{it}}{Q_t} = \left(\frac{p_{it}}{P_t}\right)^{-\theta}, \tag{5}$$

where q_i and p_i are the firm's quantity sold and price, and Q and P are the aggregate quantity and price. The firms maximize the discounted present value of profits,

$$E\left[\sum_{t=0}^{\infty} \beta^t \left(q_{it}p_{it} - \alpha q_{it}w_t - \tfrac{1}{2}\psi(p_{it} - p_{i,t-1})^2\right)\right], \tag{6}$$

where α is the labor requirement per unit of output, w_t is the nominal wage, and ψ is a parameter that determines how important the price adjustment costs are. The firm chooses its own quantity and price, taking aggregate price and quantity as given and recognizing the demand curve constraint (5). Show how to derive a New Keynesian Phillips curve from the linearized constraints and optimality conditions of this model.

There were three important elements to an answer to this question: i) the first-order conditions; ii) the insight that because these are identically situated firms they all do the same thing at the same time and therefore $p_{it} = P_t$ in equilibrium; and iii) some indication of what a "New Keynesian Phillips curve" should look like. Most exams got (i), quite a few got either (ii) or (iii), but not very many had all three elements of a good answer.

Substituting the demand curve into the objective function gives us

$$E\left[\sum_{t=0}^{\infty}\beta^t\left(\frac{p_{it}^{1-\theta}Q_t}{P_t^{-\theta}} - \alpha\frac{p_{it}^{-\theta}Q_t w_t}{P_t^{-\theta}} - \tfrac{1}{2}\psi(p_{it}-p_{i,t-1})^2\right)\right].$$

The firm's FOC is then

$$(1-\theta)\frac{Q_t p_{it}^{-\theta}}{P_t^{-\theta}} + \theta\alpha\frac{p_{it}^{-\theta-1}Q_t w_t}{P_t^{-\theta}} = \psi(p_{it}-p_{i,t-1}) + \beta\psi E_t[p_{i,t+1}-p_t].$$

Using the equilibrium condition that $p_{it} = P_t$, we then get

$$(1-\theta)Q_t + \theta\alpha\frac{Q_t w_t}{P_t} = \psi(P_t - P_{t-1}) + \beta\psi E_t[P_{t+1}-P_t].$$

In a steady state with constant price level, the left and right hand sides of the expression above are both zero, which means that when we linearize the system, Q drops out. (Most exams did not notice this. Correctly deriving the more complicated Phillips-curve-like equation that leaves Q effects in could get full credit.) Using π_t to represent $\log P_t/P_{t-1}$ as a deviation from zero, we therefore arrive at

$$\pi_t = \beta E_t\pi_{t+1} + \frac{\bar{Q}\theta\alpha}{\psi\bar{P}}\omega_t\,,$$

where ω_t is the deviation from steady state of the real wage w_t/P_t. This is the standard form of a new Keynesian Phillips curve — current inflation is the discount rate times expected inflation next period plus a term reflecting marginal costs. The $\bar{P}$ term here is not standard, and reflects my having made the adjustment costs quadratic in the level of price changes rather than in their log or percentage changes.