EXERCISE DUE TUESDAY, 4/13

Consider a representative agent AK growth model with a capital tax, in which there is a government expenditure at the initial date $t = 0$ that can be financed only with a capital tax, though the timing can be adjusted by issuing debt. The agent solves

$$\max_{C,K,B} \sum_{t=0}^{\infty} \beta^t \log C_t \qquad \text{subject to} \tag{1}$$

$$C_t + K_t(1 + \tau_t) + B_t = R_{t-1}B_{t-1} + AK_{t-1}\,. \tag{2}$$

Assume $A > 1$ and $0 < \beta < 1$. The government budget constraint is

$$B_t + \tau_t K_t = R_{t-1}B_{t-1} + g_t\,. \tag{3}$$

We assume $B_{-1} = 0$, $g_0 > 0$ and $g_t = 0$ for $t > 0$.

To make the problem straightforward, there should have been a "$+g_t$" on the right-hand side of the private budget constraint. That is, the problem should have made the expenditure a transfer payment that did not affect total resources available for consumption and investment. Without that modification, introduction of the capital tax is tied to a g_0 that is an actual drain on resources, so that increasing τ, even in the neighborhood of zero, reduces welfare because of the implied increase in g_0.

The answers below are given for the problem modified to add g_0 on the right-hand side of the budget constraint. To make sense of the problem with g_0 a resource drain, one has to consider g_0 held fixed while changing among different types and time paths of taxes that might fund the same g_0. This does not really change the analysis, though, since the effect of recognizing a given resource-draining g_0 is equivalent to starting the economy with a smaller capital stock K_{-1}. In all the discussion below, one could replace AK_{-1} by $AK_{-1} - g_0$ and interpret the various tax schemes as ways to replace a lump-sum tax financing the initial expenditure.

(1) Consider a government policy that finances g_0 by instituting a permanent, constant capital tax $\tau_t \equiv \bar{\tau}$. This will of course involve financing part of the initial spending with debt issuance. Check whether, as argued in class, the derivative of the agent's objective with respect to $\bar{\tau}$ is zero at $\tau = 0$. (Note that this assumes we are in the neighborhood of $g = 0$, since $\bar{\tau}$ will be determined by g_0.)

With a constant tax, the FOC's take the form

$$\frac{1+\bar{\tau}}{C_t} = \frac{\beta A}{C_{t+1}} \tag{4}$$

$$\frac{1}{C_t} = \frac{\beta R_t}{C_{t+1}}\,, \tag{5}$$

so C grows at the rate $A\beta/(1+\bar{\tau})$ and $R_t \equiv A/(1+\bar{\tau})$. (Note that this is real debt, so there is no price level and the government must match the real yield on bonds to that on capital.) The social resource constraint in this case is

$$C_t + K_t = AK_{t-1}\,,$$

Solving forward, we arrive at

$$K_{-1} = \sum_{t=0}^{\infty} A^{-t-1} C_t = A^{-1} \sum_{t=0}^{\infty} C_0 \beta^t (1+\bar{\tau})^{-t} = A^{-1} \frac{C_0(1+\bar{\tau})}{1+\bar{\tau}-\beta}$$

or, rearranging

$$C_0 = \frac{(1+\bar{\tau}-\beta)}{1+\bar{\tau}} AK_{-1}\,.$$

Thus we can see that the effect of the tax is to increase initial consumption, while it lowers the growth rate of consumption.

The objective function value can then be calculated directly as

$$\sum_{t=0}^{\infty} \beta^t \Big(\log C_0 + t\big(\log A + \log\beta - \log(1+\bar{\tau})\big)\Big)$$
$$= \frac{1}{1-\beta}\log C_0 + \sum_{t=0}^{\infty} \beta^t \Big(t\big(\log A + \log\beta - \log(1+\bar{\tau})\big)\Big)\,.$$

We can calculate

$$\frac{\partial \log C_0}{\partial \bar{\tau}} = \frac{\beta}{(1+\bar{\tau})(1+\bar{\tau}-\beta)}$$

$$\frac{\partial \log(1+\bar{\tau})}{\partial \bar{\tau}} = \frac{1}{1+\bar{\tau}}$$

and use these facts to calculate the derivative of the objective function w.r.t. $\bar{\tau}$ as

$$\frac{\beta}{(1-\beta)(1+\bar{\tau}-\beta)(1+\bar{\tau})} - \sum_0^\infty \beta^t \frac{t}{1+\bar{\tau}}\,.$$

Then we can use the fact that

$$\sum_0^\infty t\beta^t = \frac{\beta}{(1-\beta)^2}$$

to simplify further and arrive at

$$\frac{-\beta\bar{\tau}}{(1-\beta)^2(1+\bar{\tau}-\beta)}$$

as the derivative of the objective function. It is easy to see that this is zero when $\bar{\tau} = 0$.

Note that a transfer financed as a pure lump-sum tax would have no welfare effect at all, since agents would care only about the net amount of transfer minus tax, which would be zero.

(2) Consider a policy that instead finances the entire expenditure with a one-time capital tax $\tau_0 > 0$, with no debt issuance at all (and therefore $\tau_t = 0$ for $t > 0$). Check whether the derivative of the objective function with respect to this tax is zero in the neighborhood of $\tau_0 = 0$.

Now the growth rate will not be affected, except in the first period. The tax will simply reduce investment in the first period. We will have $C_1/C_0 = A\beta/(1+\tau_0)$ and $C_t/C_{t-1} = A\beta$ for $t > 1$. We can still solve forward to find C_0 as before:

$$K_{-1} = A^{-1}C_0 + A^{-2}\sum_{t=0}^{\infty} C_1\beta^t = C_0A^{-1}\left(1 + \frac{\beta}{(1-\beta)(1+\tau_0)}\right)$$

$$\therefore C_0 = AK_{-1}\frac{1-\beta+\tau_0(1-\beta)}{1+\tau_0(1-\beta)} .$$

We can see again that initial consumption increases, while the first-period growth rate of consumption is decreased.

The objective function value in this case is

$$\log C_0 + \beta\sum_{t=0}^{\infty}\beta^t\left(\log C_0 + \log(A\beta) - \log(1+\tau_0) + t\log(A\beta)\right)$$

$$= \frac{\log C_0}{1-\beta} + \frac{\beta}{1-\beta}\log\left(\frac{A\beta}{1+\tau_0}\right) + \left(\frac{\beta}{1-\beta}\right)^2\log(A\beta)$$

The derivative of the objective function w.r.t. τ_0 is therefore

$$\frac{-\tau_0\beta}{(1+\tau_0)(1+\tau_0(1-\beta))} .$$

Again, it is easily seen that this is zero at $\tau_0 = 0$.

(3) Compare the agent objective function under the two financing schemes. Which is better?

Probably the best way to do this is to consider the local behavior around zero tax, and look at second derivatives. We already have, above, the first derivatives of the objective function w.r.t. the taxes. We can find the second derivatives at $\tau = 0$ as

$$\partial\tau_0^2 : \qquad -\beta$$
$$\partial\bar{\tau}^2 : \qquad \frac{-\beta^2}{(1-\beta)^3}\,.$$

The required tax for a given transfer g_0 will be larger for the lump sum tax, so we have to multiply these two second derivatives by $d\tau_0/dg_0$ and $d\bar{\tau}/dg_0$, respectively to get the second derivative of the objective function with respect to the financing need. Evaluating these derivatives with respect to g_0 at $g_0 = 0$ is easy, and leads to

$$\frac{dg_0}{d\tau_0} = \beta AK_{-1}$$
$$\frac{dg_0}{d\bar{\tau}} = \frac{AK_{-1}}{\beta - \beta^2}\,.$$

So the losses increase faster with g_0 for the constant capital tax so long as

$$\frac{\beta^3}{(1-\beta)^2} > 1\,.$$

For discount factors β in the usual range, this is true, though if $\beta < .569$ it is not.

(4) Derive the Euler equation first order conditions for the optimal time path of τ_t, and check whether either of the two simple policies discussed above satisfies the FOC's. (I think neither does. Solving the FOC's and constraints for the optimal path of τ would be a mess, probably, but showing that some implication of the FOC's is violated by these policies should be easier.)

The problem now has to take the private FOC's as constraints, so it can be expressed as

$$\max_{C,K,\tau,B} \sum_{t=0}^{\infty} \beta^t \log C_t \quad \text{subject to}$$
$$\lambda : \qquad C_t + K_t = AK_{t-1}$$
$$\mu : \qquad B_t = \frac{A}{1+\tau_{t-1}} B_{t-1} + g_t - \tau_t K_t$$
$$\nu : \qquad \frac{1+\tau_t}{C_t} = \beta \frac{A}{C_{t+1}}\,,$$

where we have simplified by setting $R_t = A/(1+\tau_t)$ and therefore omitting the private bond FOC.

The government's Euler FOC's are

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t - \frac{1}{C_t^2}(\nu_t(1+\tau_t) - A\nu_{t-1})$$

$$\partial K: \qquad \lambda_t = \beta A \lambda_{t+1} - \tau_t \mu_t$$

$$\partial B: \qquad \mu_t = \beta \mu_{t+1} \frac{A}{1+\tau_t}$$

$$\partial \tau: \qquad \mu_t K_t + \frac{\nu_t}{C_t} = -\beta \frac{A}{(1+\tau_t)^2} B_t \mu_{t+1}\,.$$

If τ is constant, the private budget constraint, the private C FOC, and the private K TVC tell us that the growth rate of consumption is constant at $A\beta/(1+\bar{\tau})$, and that the steady state value of $\kappa = K/C$ is fixed. The government budget constraint, divided through by C_t, is then an unstable difference equation in B/C whose only solution that satisfies the B private TVC makes $b = B/C$ fixed. Thus B, C, and K all grow at the same rate $A\beta/(1+\bar{\tau})$. The government's B FOC tells us that μ shrinks at the rate $(1+\bar{\tau})/(A\beta)$. Then the government's τ FOC tells us that ν/C is a constant. The government's C FOC, multiplied by C_t, then tells us that $\lambda_t C_t$ is a constant. However, the government's C FOC is not that shown above when $t = 0$, because at $t = 0$ the lagged private C FOC does not constrain the government, and therefore $\nu_{-1} = 0$. Thus our calculation of λC will give a different answer at $t = 0$ from what we get at $t > 0$. This will make it impossible to satisfy the government's K FOC. That is, we can calculate $\lambda_0 C_0$ from the K FOC as

$$\lambda_0 C_0 = (1+\bar{\tau})\lambda_1 C_1 - \bar{\tau}\mu_0 C_0$$

and the result must be, since all values of $\lambda_t C_t$ are constant for $t > 0$ and $\mu_t C_t$ is constant for all t, including $t = 0$, the same constant value for $\lambda_0 C_0$ that we have for $\lambda_t C_t$ when $t > 0$. but since ν_t / C_t is constant for all t, this contradicts the time 0 government C FOC unless $\nu_t = 0$, all t. But if $\nu_0 = 0$, the τ and B government FOC's imply an exact relation between K_0 and B_0 that contradicts the first period government budget constraint — unless also μ_t is identically zero. But if both μ and ν are always zero, the government's C FOC says $\lambda_t = 1/C_t$, and the government's K FOC contradicts the private K FOC whenever $\bar{\tau} > 0$.

For the case of a one-time capital tax, there is an easier indirect argument. From time $t = 1$ onward, the economy is in the optimal, no-tax equilibrium, given whatever initial capital has been left to it by time-0 policy. But we know that the marginal effect on welfare of a small capital tax in the neighborhood of zero tax — either a one-time tax at $t = 1$ or a permanent tax from $t = 1$ onward — is zero. Since there is a positive tax τ_0 at $t = 0$, the marginal effect of reducing that tax is non-zero. Hence it must be optimal to have some taxation at dates beyond $t = 0$.

(5) Would it affect your answers if we eliminated the $(1 + \tau_t)$ factor in front of K_t on the left-hand side of the budget constraint and instead replaced AK_{t-1} on the right-hand side by $AK_{t-1}/(1 + \tau_t)$?

Yes. In that case a one-time capital tax would behave just like a lump-sum tax, since it would have no effect on the private rate of return on capital at any future date. In other words, the base for the tax, AK_{-1}, is completely inelastic, being given by history, and hence a tax on it is non-distorting. The permanent capital tax, on the other hand, would still be distorting.