## MID-TERM EXAM

*This is a 90-minute exam. Answer all three questions, each of which is worth 30 points. You can get partial credit for partial answers. Do not spend disproportionate time on any one question unless you have already answered the others.*

(1) In an overlapping generations model with a constant population of two-period-lived agents we have the agents in the generation born at $t$ solving

$$\max_{C_1, C_2, M, K} \log(C_{1t} C_{2,t+1}) \qquad \text{subject to}$$
$$C_{1t} + \frac{M_t}{P_t} + K_t + \bar{\tau} = \bar{Y}$$
$$C_{2,t+1} = \frac{M_t}{P_{t+1}} + AK_t\,.$$

The government budget constraint, in per capita terms, is

$$\frac{M_t}{P_t} = \frac{M_{t-1}}{P_t} - \bar{\tau}\,,$$

that is, the government uses the tax proceeds each period to buy money. Assume $A > 1$.

(a) Show that, if certain constraints on $\bar{\tau}$ are satisfied, this economy has a unique, positive equilibrium initial price level, and that in this equilibrium $C_{1t}$ and $C_{2,t}$ are constant across generations $t$.

It is natural to assume (and the problem should have so stated) that $K_t \geq 0$ and $M_t \geq 0$ are required. Assuming these constraints are not binding, the FOC's are

$$\begin{aligned}
\partial C_1 :& \qquad \frac{1}{C_{1t}} = \lambda_t \\
\partial C_2 :& \qquad \frac{1}{C_{2,t+1}} = \mu_{t+1} \\
\partial M :& \qquad \frac{\lambda_t}{P_t} = \frac{\mu_{t+1}}{P_{t+1}} \\
\partial K :& \qquad \lambda_t = A\mu_{t+1}\,.
\end{aligned}$$

These can be reduced to

$$C_{1t}P_t = C_{2,t+1}P_{t+1}$$
$$\frac{C_{2,t+1}}{C_{1t}} = A = \frac{P_t}{P_{t+1}}\,.$$

The social resource constraint, derivable from the government constraint and the private constraints, is

$$\bar{Y} - K_t - C_{1t} = AK_{t-1} - C_{2t}\,.$$

So long as both money and capital are being held in non-zero amounts, the FOC's tell us that they must have the same rate of return, $A$, and in this model with no uncertainty, they appear to the young at date $t$ as equivalent assets. The young's decisions about how much to save therefore do not depend on how they divide their savings between $M$ and $K$. In fact the young see their constraint as

$$C_{1t} + \frac{C_{2,t+1}}{A} = \bar{Y} - \bar{\tau}\,, \tag{$*$}$$

from which we can conclude that $C_{1t} = (\bar{Y} - \bar{\tau})/2$ and $C_{2,t+1} = A(\bar{Y} - \bar{\tau})/2$ for all $t \geq 0$.

The GBC and the requirement that $P$ shrink at the rate $A$ imply that

$$\frac{M_{t+1}}{P_{t+1}} = A\frac{M_t}{P_t} - \bar{\tau}\,.$$

This is an unstable difference equation. Since the amount of savings each period in real terms is constant, $M/P$ cannot become arbitrarily large. And if $M/P$ falls below steady state the GBC implies it will eventually become negative, which is impossible. (And on such a path private agents would see that they would run out of money and would be unable to pay their tax obligations.) So the only possible value of $M/P$ is the steady state solution of this equation:

$$\frac{M_t}{P_t} \equiv \frac{\bar{\tau}}{A-1}\,.$$

However, if $\bar{\tau}$ is too large, the steady state value of $M/P$ might exceed the total savings of the young, and this would be impossible. So $\tau$ must be between 0 and the value that makes the steady state $M/P$ equal to total saving.

The initial value $M_{-1}$ is inherited from the past, so, recognizing that $M_0/P_0$ must match the unique steady state value, the GBC at time 0 implies

$$\frac{\bar{\tau}}{A-1} = \frac{M_{-1}}{P_0} - \bar{\tau}\,,$$

which implies the unique solution

$$P_0 = \frac{A-1}{A\bar{\tau}} M_{-1} \,.$$

(b) Show that in this equilibrium the constant utility enjoyed by each generation is lower than the level of utility that would prevail in steady state if there were no money and $\bar{\tau} = 0$.
From the private budget constraint $(*)$ we can see that the consumer's effective budget set shrinks with $\tau$, so that utility in steady state is lowered for all generations $t \geq 0$. If the equilibrium has prevailed forever, then this result applies to all generations.

(c) Suppose the economy has been in the equilibrium with $\tau > 0$ and valued money up until time $T$, and everyone in the economy believed this equilibrium would persist forever. However, at time $T$ the government surprises the public by deciding to switch to a policy of $\bar{\tau} = 0$ at date $T$ and forever thereafter. Would this be a Pareto improvement? (I.e., would it make everyone better off?) Explain your answer.
Short answer: no. All generations born at $T$ or later are indeed better off. But those in generation $T-1$ are worse off. They have savings in the form of $K_{-1}$ and $M_{-1}$. They thought this was going to provide them with consumption $C_{2T} = A(\bar{Y} - \bar{\tau})/2$ at $t = T$, but to their surprise their money savings have become valueless, so their consumption is lower than it would have been if the old equilibrium had persisted.

(2) A government wishes to maximize

$$-E\left[\sum_{t=0}^{\infty} \tfrac{1}{2}(\theta u_t^2 + \pi_t^2)\beta^t\right]$$

subject to

$$u_t = \bar{u} - \phi(\pi_t - \mathcal{E}_{t-1}\pi_t) + \gamma u_{t-1} + \nu_t \,,$$

where $u$ is unemployment, $\pi$ is inflation, and $\mathcal{E}$ is the expectation operator used by the public — possibly not ordinary mathematical expectation.

(a) Suppose $\mathcal{E}_{t-1}\pi_t = \pi_{t-1}$, that is, the public always expects next period's inflation to match this period's. Derive first order conditions for an optimum in this case. Are transversality conditions an issue here? If so, what are they? Do you need to assume that the government can commit to future actions and be believed in order to justify your equations?
The Euler equation FOC's are

$$-\theta u_t = -\lambda_t + \beta\gamma E_t\lambda_{t+1} - \pi_t = -\phi\lambda_t + \beta\phi E_t\lambda_{t+1} \,,$$

from which we can derive

$$\lambda_t = \frac{\gamma\pi_t - \theta u_t}{\phi\gamma - 1}\,.$$

Getting correct TVC's here is tricky. The problem statement suggests the constraint is exact, and give no constraints on the signs of $\pi$ or $u$. Thus to get the problem in the form of our standard setup from the notes, the constraint should be replaced with two inequalities and two corresponding positive Lagrange multipliers. In this case, where there is only one equality constraint, one can equivalently work with just one multiplier, but remember that it can change sign. A sufficient condition for the TVC (a $\lim$ that implies the two $\liminf$'s in the full TVC) is then

$$\lim_{t\to\infty} \beta^t E[(\lambda_t - \theta u_t)(\bar{u}_t - u_t) + (\phi\lambda_t - \pi_t)(\bar{\pi}_t - \pi_t)] = 0$$

or equivalently

$$\lim_{t\to\infty} E\big[\big(\phi(\bar{\pi}_t - \pi_t) + \gamma(\bar{u}_t - u_t)\big)\lambda_{t+1}\big] = 0\,,$$

where $\bar{u}_t$ and $\bar{\pi}_t$ are candidate paths other than the solution paths $u_t, \pi_t$. In the second form this TVC implies that the two FOC's can be solved forward, but you were not asked to pursue the solution here.

There are no expectational constraints here, so the issue of government commitment does not arise.

(b) Suppose $\mathcal{E}_{t-1}\pi_t = E_{t-1}\pi_t$, that is, the public has rational expectations. Again derive first order conditions for an optimum, consider transversality issues, and explain whether you need to assume the government can commit to future actions and be believed.

First we have to introduce a new variable $z_t = E_t\pi_{t+1}$, and use this definition as a new equation in the system. Now the Euler equation FOC's are

$$\begin{aligned}
\partial u_t: &\qquad -\theta u_t = -\lambda_t + \beta\gamma E_t\lambda_{t+1}\\
\partial \pi_t: &\qquad -\pi_t = -\phi\lambda_t + \beta^{-1}\mu_{t-1}\\
\partial z_t: &\qquad \beta\phi E_t\lambda_{t+1} = \mu_t\,.
\end{aligned}$$

Here the TVC is

$$\beta^t E\big[\lambda_{t+1}\big((\bar{\pi}_{t+1} - \pi_t)\phi + (\bar{u}_t - u_t)\gamma\big)\big] = 0\,.$$

This solution does suppose that the government can commit and be believed. This is reflected in the fact that in the first period, the $\pi$ FOC takes on a different form:

$$-\pi_t = -\phi\lambda_t\,.$$

(c) What equations characterize Woodford's "timeless perspective" solution in this case.?
Woodford's timeless perspective solution is obtained by just using the steady-state equations even in the first period.

(d) If the government cannot commit, so that it simply takes the policy rule of future periods' governments as given, will it in this model always choose the same action that would be chosen in the first period by a government that can commit?
No. This was true in our classroom Barro-Gordon style model, but that was because there was no time-dependence in the solution. Everything emerged as i.i.d. Here the lagged unemployment term is a state variable. Though the non-committing policy maker takes the decision rules of future policy makers as given, he does recognize that his current policies affect the state (here $z_t$ and $u_t$) that next period's policy maker witll inherit, and that this will affect future utility values. This is reflected here in the fact that the FOC's involve expected future values of $\lambda_t$.. You were not expected to derive even equations for the behavior of the policy-maker here, just to recongize that the non-committing policy-maker will neither produce the full commitment solution nor reproduce each period the behavior of the

(3) (a) Leeper concludes that when fiscal policy is active and monetary policy passive, or vice versa, there is a uniquely determined price level. When both policies are active, he concludes there is no equilibrium, and when both are passive, he concludes that the initial price level is indeterminate. In class and in some readings, we encounter models where these propositions fail. Explain why.
Leeper works only with linear models, and assumes that exponentially explosive paths of the economy are not equilibria. What we did in class was to specify complete nonlinear models, which we solved analytically. In some of these models explosive paths could not be ruled out as equilibria, which made Leeper's framework not apply.

(b) Chamley shows that optimal capital taxes tend to zero in the long run. Judd, in a classic paper (not on the reading list and not discussed directly in class) shows that it is optimal for capital taxes not to go to zero, but instead to oscillate randomly forever. From principles we have discussed in class, explain how Judd might have reached this conclusion and why it differs from Chamley's.
Chamley argues in a non-stochastic model. When fiscal needs evolve stochastically over time, revenue must somehow adjust. Capital taxes are distorting only because they are anticipated, and hence distort investment. If capital taxes can be adjusted to respond to fiscal shocks, but with the

capital taxes always having a priori mean zero, they are completely non-distorting. So Judd's result is that they should be zero on average, but constantly fluctuating. Of course this is not a practical suggestion. However, a similar argument supports the idea of keeping primary surpluses nearly constant and absorbing unanticipated fiscal shocks via unanticipated inflation.

(c) In class, and in a paper by Diamond, we saw examples of models in which postponing taxation by issuing public debt can shift a tax burden from one generation to another. Would such burden-shifting be possible in an economy without any capital? Explain why or why not.
An acceptable answer is "no", even though this is not quite correct. Both our in-class model and Diamond's more general setup shift the fiscal burden by reducing investment, and this mechanism is not available in an economy without capital. Without capital, any resources absorbed by government spending must come out of current consumption of the old and young. If the agents have log utility, they consume in the first period of life half of what they perceive as their total income, with the future discounted at the rate reflecting their first and second period endowments. They will buy debt only if it appears to them to increase their overall discounted income, and if it does, they will *increase* current period consumption. So there will be no equilibrium in which the government raises real resources by selling debt. However, if agents have, say $C^{1-\gamma}/(1-\gamma)$ (CRRA) utility with $\gamma < 1$, the government can induce private agents to reduce current-period consumption by offering a high enough real return on government debt. If next period the government wants to further postpone taxes, it will have to issue new debt to pay off the old debt, and will again have to offer a high yield to sell the debt. In this way it could postpone taxation, even without capital.