NOTES ON CONTINUOUS TIME SEARCH EQUILIBRIUM

The basic continuous time search model we worked out in class can be handled without the "second-order approximation" hand waving I used in the lecture.

Notation:

- $F()$: cdf of wage offers
- w: an actual wage offer
- $W(w)$: present value of wages, after acceptance of wage offer w
- b: dollar value per unit time of benefits in unemployed state
- U: present value of wages when unemployed and without a current wage offer
- α: rate of arrival of job offers when unemployed
- λ: rate of arrival of job terminations when employed
- r: rate of time discount

The probability of getting no job offer over an interval $(0, \delta)$ is $e^{-\alpha\delta}$. This can be taken as the definition of job offers arriving at the rate α per time unit. The cdf for the arrival time of the first offer is then obviously $1 - e^{-\alpha t}$, which in turn implies that the pdf of the arrival time is $\alpha e^{-\alpha t}$. Note that this distribution implies 1) that the probability of a job offer over arriving over an interval $(0, \delta)$ is approximately $\alpha\delta$, so α can be interpreted as the rate per unit time at which job offers arrive, and 2) the conditional distribution of time to next offer given that none has arrived so far (found by truncating the pdf on the left at the current value of t and renormalizing it to integrate to 1) is again $\alpha e^{-\alpha t}$, so that offer arrivals are independent across time.

The probability of a job not terminating over an interval $(0, \delta)$ is $e^{-\lambda\delta}$. If a worker is employed at $t = 0$ in a job that will pay w per unit time until it ends, and if we knew that the job ended at $t = T$, the present value of the wage stream would be

$$\int_0^T e^{-rs} w \, ds + e^{-rT} U \,. \tag{1}$$

That is, it would be the present value of the wage over the time the job lasts, plus the present value of the unemployed state that begins at T. Taking the expectation over T of the present values conditional on T in (1) gives us

$$\begin{aligned}W(w) = \int_0^\infty \lambda e^{-\lambda T} \int_0^T e^{-rs} w\, ds\, dT + \int_0^\infty \lambda e^{-(\lambda+r)T} U dT \\ = \int_0^\infty \lambda e^{-\lambda T} \frac{1-e^{-rT}}{r} w\, dT + \frac{\lambda U}{\lambda + r} \\ = \left(\frac{\lambda}{r\lambda} - \frac{\lambda}{r(r+\lambda)} \right) w + \frac{\lambda U}{\lambda + r} \\ = \frac{w}{r+\lambda} + \frac{\lambda U}{r+\lambda} \,. \qquad (2)\end{aligned}$$

This then can be rearranged as the same equation we had in lecture:

$$(r+\lambda)W(w) = w + \lambda U \,. \qquad (3)$$

The same type of derivation works for finding U. If we knew that the first job offer would arrive T periods from now (time $t = 0$), we would have

$$U = \int_0^T b e^{-rs} ds + e^{-rT} \int \max\{W(w), U\}\, dF(w) \,. \qquad (4)$$

The pdf of the arrival time of job offers is $\alpha e^{-\alpha T}$. Taking expectations over T gives us

$$\begin{aligned}\int_0^\infty b\alpha e^{-\alpha T} \frac{1-e^{-rT}}{r} dT + \frac{\alpha}{r+\alpha} \int \max\{W(w), U\}\, dF(w) \\ = \frac{b}{r} - \frac{b\alpha}{r(r+\alpha)} + \frac{\alpha}{r+\alpha} \int \max\{W(w), U\}\, dF(w) \\ = \frac{b}{r+\alpha} + \frac{\alpha \int \max\{W(w), U\}\, dF(w)}{r+\alpha} \,, \qquad (5)\end{aligned}$$

which can be rearranged to match the equation we derived in class

$$rU = b + \alpha \int \max\{W(w) - U, 0\}\, dF(w) \,. \qquad (6)$$

From (3) we can see that there will be a **reservation wage**. U does not depend on the current wage (even though it does depend on the distribution of wage offers), so the right-hand side of (3) is increasing in w. The criterion for accepting a job offer is $W(w) > U$. Therefore there will be a $\bar{w}$ such that job offers at $w > \bar{w}$ will be accepted and offers at a wage lower than that will not be accepted.

In a steady state equilibrium, the rate at which workers leave jobs must match the rate at which they are being hired. Let u be the unemployment rate, the fraction of the work force that is unemployed. Then in steady state we must have

$$u\alpha P[w > \bar{w}] = (1-u)\lambda\,, \text{ i.e.} \tag{7}$$

$$u = \frac{\lambda}{\lambda + \alpha(1 - F(\bar{w}))}\,. \tag{8}$$

From (6) we can see that U must be increasing in b. The integral on the right has an integrand that is weakly decreasing in U, so U itself must be increasing in b. $\bar{w}$ is clearly increasing in U, from which we conclude, looking at (8), that unemployment is increasing in b.

Note that the conclusion that unemployment is increasing in b has no welfare implications in this model. The model does not explain where the resources to pay b come from. If they are free, the optimum in the model is to raise b as high as possible, so no one has to work (though in this model we have also not so far given people any reason to dislike working) and yet have very high incomes. Interesting question: if b is financed by a lump-sum tax on those working, what is the optimal level of b?