SEARCH AND INFORMATION THEORY EXERCISES

(1) Consider the model, already studied in class, in which there is an exogenous distribution of wage offers with cdf $F(w)$, a Poisson arrival rate of job offers to the unemployed of α, a Poisson arrival rate of job terminations of λ, and a payment per unit time to the unemployed of b. Assume agents discount the future at the rate r. To make things concrete here we assume the distribution of job offers concentrates on just three points: $w = 2, 3, 4$. Every job offer has equal probability (one third) of taking on any of these values. Agents also have linear utility, so their flow of utility at any date is equal to their wage earnings or their benefit payments. Assume the numerical values for parameters given below:

w1	w2	w3	b	r	λ	α	pw1	pw2	pw3
2.00	3.00	4.00	0.00	0.05	0.02	0.20	0.33	0.33	0.33

(a) Show that the total wages being earned in the economy per unit time in steady state depends only on the exogenous parameters listed above and the reservation wage. (Assume a constant population of 1.0).
Wages of those employed are random draws from the $F()$ distribution, truncated from below at the reservation wage $\bar{w}$. The rate of exit from unemployment per unit time is $\alpha \cdot (1 - F(\bar{w}))$ (the rate of arrival of offers times the probability of accepting an offer). The number of workers taking jobs per unit time is u, the number of those unemployed, times this exit rate, i.e. $u\alpha \cdot (1 - F(\bar{w}))$. The number losing jobs per unit time is $(1 - u)\lambda$. The number losing jobs must match the number finding jobs in steady state, i.e.

$$\alpha \cdot (1 - F(\bar{w}))u = \lambda(1 - u)$$

If we are given $\bar{w}$, we can solve this equation for u. Then total wages being earned in steady state are given by

$$(1 - u)\frac{\int_{\bar{w}}^{\infty} w\, dF(w)}{1 - F(\bar{w})} .$$

(b) Find the reservation wage implied by the parameter settings above.
The discounted utility for someone who is unemployed without a job offer and with a reservation wage $\bar{w}$ is a constant, U, that satisfies (by arguments we went through in class and in notes)

$$(r + \alpha(1 - F(\bar{w}))U = b + \alpha \int_{w>\bar{w}} W(w) - U\, dF(w) , \qquad (\dagger)$$

where $W(w)$ is discounted utility for someone employed at the wage w. $W(w)$ satisfies

$$(r+\lambda)W(w) = w + \lambda U\,. \qquad (\ddagger)$$

The reservation wage $\bar{w}$ is that at which an unemployed person offered a job at $\bar{w}$ is indifferent between taking the job and remaining unemployed, i.e. $W(\bar{w}) = U$, so it will be $\bar{w} = rU$.

In this problem with just three values of w, we can solve the problem by breaking it into three cases, according to whether $\bar{w}$ is less than 2, between 2 and 3, or between 3 and 4. Within each of these cases, we can solve a system of linear equations for $x = (W(2), W(3), W(4), U)$. In each case we can check whether the implied $\bar{w}$ is consistent with our assumption of what interval $\bar{w}$ lies in. The four linear equations are

$$\begin{aligned}(r+\lambda)W(1) &= 2 + \lambda U \\ (r+\lambda)W(2) &= 3 + \lambda U \\ (r+\lambda)W(3) &= 4 + \lambda U \\ (r+q_j\alpha)U &= \alpha \sum p_j(i)W(i)\,.\end{aligned}$$

The last row has four different forms, according to which case we are in. With $\bar{w}$ between 3 and 4, for example, so offers at or below 3 are refused, while offers at or above 4 are accepted (case $j = 3$) we will have $q_3 = 1/3$, $p_3 = (0, 0, -\alpha/3)$. In matrix form, the equations are

$$\begin{bmatrix}(r+\lambda)I & -\lambda\mathbf{1} \\ -p_j & r+\alpha q_j\end{bmatrix} = \begin{bmatrix}2\\3\\4\\0\end{bmatrix},$$

where the three cases for the last row are

$$\begin{matrix}-.2/3 & -.2/3 & -.2/3 & .25 \\ 0 & -.2/3 & -.2/3 & .4/3+.05\,. \\ 0 & 0 & -.2/3 & .2/3+.05\end{matrix}$$

Solving this system three times, we find that the only case that produces a consistent $\bar{w}$ is $j = 2$, i.e. the case in which job offers at wages greater than two are accepted. In that case the reservation wage is 2.295.

(c) Show that this equilibrium reservation wage does not maximize the total wages being earned per unit time in steady state.

Using the formulas we derived above for steady-state u and total wages as a function of $\bar{w}$, we see that the total wages earned per unit time with the reservation wage between 2 and 3 as in the equilibrium we have solved for are 3.0435. If the reservation wage were instead between 3 and 4, the total wages per unit time are 3.0769, which is higher.

(d) Does this result suggest that optimal policy in this model will set $b > 0$? Explain your answer. (Your answer can be yes, no, or maybe — but you must explain.)

This model has no search externalities and no other way for agents' decisions to interact, so if we take the individuals' discount rate r as a correct reflection of their evaluation of current vs. future wages, the equilibrium with $b = 0$ must be better than any equilibrium that distorts their evaluation of jobs by taxing or subsidizing labor earnings or unemployment. Any transition from an equilibrium with $b = 0$ to one that raised the reservation wage would initially impact only those unemployed, who would stay unemployed longer. The eventual rise in average wages would not compensate, in terms of discounted present value, this initial period of sacrifice.

An argument for $b > 0$ within this model's linear utility, no-capital framework would have to be based on agents' being mistaken in their choices of r. While it is realistic to suppose that the unemployed have difficulty borrowing, and thus that they have distorted, high, effective discount rates, this comes from supposing that they have curved utility functions, so that low consumption, without access to borrowing, leads to a high discount rate. With linear utility, there is no connection of low current consumption to a high discount rate.

(2) Consider a monopolist whose cost per unit of output each period is x_t, an i.i.d. random variable with a pdf

$$.25(2x + x^2)e^{-x} \tag{$*$}$$

His demand curve is $q_t = 4 - p_t$ each period. He has finite Shannon capacity, so he is concerned to keep down the mutual information between his actions (choice of p_t) and the exogenous variable he is tracking (x_t).

(a) Determine how the monopolist would set p_t as a function of x_t in the absence of an information constraint.

Profits are $(4 - p)(p - x)$, so optimal p is given by $p = 2 + x/2$.

Suppose the monopolist gets a signal z_t each period that takes on only two values, 0 and 1, standing for low and high demand. Suppose also that

$$\{x_t \mid z_t = 0\} \sim xe^{-x}, \qquad \{x_t \mid z_t = 1\} \sim \tfrac{1}{2}x^2e^{-x}$$

(From this you can deduce that z must be 0 or 1 with equal probability.)

(b) Determine the mutual information between z_t and x_t.

The mutual information between two variables is the expected reduction in entropy in either one of them, conditional on the other, where entropy is defined as

$\int p(x)\log(p(x))\,dx$. The pre-signal entropy of x (in nats, rather than bits, meaning we're using log base e) is

$$-\int_0^\infty .25(2x+x^2)e^{-x}(-2\log(2)+\log(2x+x^2)-x)\,dx = 1.7704\,.$$

For the conditional distribution of $x \mid z = 0$ we get 1.5772 nats, and for the conditional of $x \mid z = 1$ we get 1.8476 nats. The average conditional entropy is therefore 1.7124 nats, and the average reduction of entropy is .0580 nats, or .0837 bits. (bits=nats$\times \log_2 e$) Note that this is a case where one of the signals, $z = 1$, actually increases the entropy; we are more uncertain about x after seeing $z = 1$ than we were before we saw $z = 1$. Nonetheless, as is necessarily true, the average reduction in entropy from observing z is positive.

(c) Find the optimal choice of the function $p(z_t)$ that maps the signal z_t into the action p_t at t, assuming that the $x_t \mid z_t$ distributions are specified as above. Note that the firm may, because it observes x_t imperfectly, sometimes set p_t so that realized profits turn out to be negative.

This is a simple static linear-quadratic optimization, so certainty-equivalence applies. (There are no constraints, and the objective function is quadratic jointly in the choice variable p and the random variable x.) That is, the stochastic problem can be solved by substituting expectations for unknown random quantities (here x). We have $E[x \mid z = 0] = 2$, since this pdf is a $\text{Gamma}(2)$ distribution, and $E[x \mid z = 1] = 3$, since this pdf is a $\text{Gamma}(3)$ distribution. Thus we can substitute into the formula above for optimal choice of p to find that with $z = 0$, optimal p is 3, and with $z = 1$ optimal p is 3.5. If we did not get any signal, the best we could do is set $p = 3.25$. The entropies here were all calculated using R's `integrate()` function. E.g. `integrate(function(x) -dgamma(x,2) * log(dgamma(x,2)),0,20)` to calculate the integral for the $x \mid z = 1$ case.

(d) Compare expected profits in this finite-information-flow solution with expected profits in the unconstrained optimal solution and with expected profits when no signal is available (i.e. it is known only that x_t has the pdf $(*)$).

Expected profits in the three cases $z = 0$, $z = 1$, and no signal, are 1, .25, and .5625, respectively. The average of profits across $z = 0$ and $z = 1$ when the information is used is .625, which exceeds what is obtained with no signal by .0625. Notice that a tiny amount of information, less than a tenth of a bit, is enough to increase expected profits by more than 10%. In the unconstrained optimum expected profits are a bit tricky to calculate. Just plugging in $p = 2 + x/2$ gives a very high expected profit. But the $p = 2 + x/2$ formula implies that for $x > 4$, quantity becomes negative. $p - x$, profit margins, also becomes negative for very large x, so that the product of the two becomes large and positive. The idea that profit margins might become negative because of a large x even with p

in (0,4) makes some economic sense. But the idea that profits can be made by selling large negative amounts at negative profit margins makes no sense. So the realistic answer for the unconstrained optimum assumes that profits and quantities are zero when $x \geq 4$, i.e. profits are

$$\int_0^4 .25(2x + x^2)e^{-x}(2 - \tfrac{x}{2})^2 \, dx = 1.099 \,.$$

To actually attain this level of profits would require infinite mutual information between p and x, however, since x is continuously distributed.

Note that in this problem all the densities are Gamma densities or linear combinations of Gamma densities, so finding their expectations analytically is possible. Evaluating their entropies may require numerical integration.