NONLINEAR MODEL-SOLVING EXERCISE

We are going to check accuracy of several approaches to solving the standard growth model with CRRA utility, no labor, and Cobb-Douglas technology. We use this model because it has a single state variable and because it has a limiting special case in which an analytic solution is available.

There is a representative agent in the model, who solves:

$$\max_{C,K} E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma}\right] \text{ subject to}$$
$$C_t + K_t = A_t K_{t-1}^{\alpha}, \quad t = 0, \ldots, \infty .$$

In the special case $\gamma = 1$, the utility function becomes $\log C_t$ and the solution makes C_t / K_t constant. We assume $\log A_t$ is i.i.d. $N(0, \sigma^2)$.

A useful fact is that if we define $W_t = A_t K_{t-1}^{\alpha}$, we can rewrite the constraint in the standard dynamic programming form

$$W_t = A_t (W_{t-1} - C_{t-1})^{\alpha}, \qquad (*)$$

in which case we know the solution will have the form $C_t = f(W_t)$ for some function f.

You are to consider the following parameter values:

$$\beta = .9$$
$$\gamma = 4, .25$$
$$\alpha = .7$$
$$\sigma^2 = .04 .$$

For each of the two values of γ listed,

(a) Find the deterministic steady state.
(b) Log-linearize about the steady state and find the corresponding decision rule. (Being log-linear, it will have the form $C_t = \theta W_t^{\phi}$ for some θ and ϕ.)
(c) Simulate and plot 5 time paths of length 100 from the model, all starting with W_0 equal to the deterministic steady state value of W, using the log-linear decision rule you derived above together with the original nonlinear state-evolution equation $(*)$.
(d) Check accuracy of your solution by estimating a regression of the simulated Euler equation errors at $t + 1$ on your five paths against two or three functions of W_t. You might choose these functions by plotting the simulated errors against corresponding W's.

(e) Check accuracy by analytically computing the expectation in the Euler equation and considering its difference from zero. Do this at a grid of values of W equal to $\bar{W}$ times .25, .5 ,.75, 1, 1.3, 2, and 3, where $\bar{W}$ is the deterministic steady state. Note that the expectation of e^z when $z \sim N(0, \nu^2)$ is $\exp(.5\nu^2)$.
(f) Using a nonlinear equation-solving routine (e.g. `csolve.m` or `csolve.R` on the course website, or the built-in routines in Matlab, Mathematica, or R), find the combination of θ and ϕ values in $C_t = \theta W_t^{\phi}$ that sets to zero the sum of discrepancies over the seven multiples of $\bar{W}$ listed above and the difference between the sum of the first three and the sum of the last three. (This is a crude version of a projection method.)
(g) Repeat with this optimized loglinear rule the plots and the two types of accuracy checks you made above for the local approximation.
(h) By using the loglinear decision rule, it is possible to write the model's Euler equation entirely in terms of current and future W_t and C_t, with no occurence of A_t. Using this form of the Euler equation and the local approximation to the decision rule (not the optimized one), construct a backsolved solution. You can get a value for the variance of the Euler equation errors from the solution of the log-linearized model. Plot the time paths of this solution, and check how far the properties of the A_t sequence you solve for are from the original specification.

Remarks:

(1) A log-linear decision rule $C = \theta W^{\phi}$ with $\theta > 0$ and $\phi > 0$ has the property that it makes $C > W$ when W gets small enough. This will make the Euler equation expression deliver NaN (not-a-number) or complex values, as they will involve taking non-integer powers of a negative number. So your programs should make the actual decision rule $C = \min\{W, \theta W^{\phi}\}$.
(2) The Euler equation used directly might have a form that allows both sides of the equation to go to zero as parameters go to extreme values. Normalize the equation to make one side of the equation equal to one before using it as a fit criterion.